Kappalab : an R package for Choquet integral based MAUT

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Introduction : context and notations

Context

Multi-criteria decision aiding (MCDA) based on multi-attribute utility theory (MAUT).

- X : a set of objects of interest;
- $N := \{1, \ldots, n\}$: a set of attributes (not necessarily cardinal), each describing a particular **aspect** of an object;
- for each $i \in N$, X_i denotes the set of values the attribute i can take;
- consequence : X can be regarded as $X_1 \times \cdots \times X_n$.

Introduction : underlying assumptions

Aim of MAUT

To model the preferences of the decision maker (DM), represented by a binary relation $\succeq$, by means of an **overall utility function** $U : X \to \mathbb{R}$ such that,

$$x \succeq y \iff U(x) \geq U(y), \qquad \forall x, y \in X,$$

where the objects x and y can be equivalently regarded as the vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively.

For U, we consider the decomposable model of Krantz et al., i.e.

$$U(x) := F(u_1(x_1), \ldots, u_n(x_n)), \qquad \forall x \in X,$$

where the functions $u_i : X_i \to \mathbb{R}$ are called the **utility functions** and F is an **aggregation function**.

Introduction : underlying assumptions

A necessary condition for the previous decomposable model to hold is that the preference relation is a **weakly separable** weak order (see e.g. Bouyssou and Pirlot 2005).

Practically, this amounts to assuming that, with each $x_i \in X_i$, the decision maker (DM) is able to associate a **measure of satisfaction** $u_i(x_i)$.

Formally, the couple (i, u_i) is called a **criterion**.

When dealing with real world MCDA problem, the determination of the utility functions can be preformed using an extension of the MACBETH approach (see e.g. Labreuche and Grabisch 2003).

In the sequel, we shall assume that the utility functions have been defined by the DM.

Introduction : the aggregation function

Classically, if **mutual preferential independence** can further be assumed, F is taken as a weighted arithmetic mean.

Independence among criteria is however rarely verified.

A suitable generalisation of the weighted arithmetic mean : the **discrete Choquet integral** w.r.t a capacity (see e.g. Marichal 2000).

Nevertheless, the use of a Choquet integral as an overall utility function clearly requires the prior **identification** of the underlying capacity.

Introduction : outline of the presentation

Aim of this presentation

- Briefly present the **Choquet integral** in the context of aggregation.
- Present the **identification** problem, i.e. how to determine the "parameters" of the Choquet integral.
- Present a toolbox, **Kappalab**, in which most of the proposed methods are implemented.

An example : the seven econometricians

Extended version of the example of Kojadinovic (2006)

7 students of an institute training econometricians evaluated in five disciplines : statistics (S), probability (P), economics (E), management (M) and English (En)

The utilities are given on a $[0, 20]$ scale

Student	S	P	E	M	En	Mean
a	18	11	11	11	18	13.80
b	18	11	18	11	11	13.80
c	11	11	18	11	18	13.80
d	18	18	11	11	11	13.80
e	11	11	18	18	11	13.80
f	11	11	18	11	11	12.40
g	11	11	11	11	18	12.40

An example : the seven econometricians

Reasoning of the DM :

- The institute is slightly more oriented towards statistics and probability;
- There are 3 groups of subjects: {statistics, probability}, {economics, management} and {English};
- Within the two first groups, subjects are somewhat substitutive, i.e. they overlap to a certain extent;
- If a student is good (bad) in statistics or probability, it is better that he/she is good in English (E or M) rather than in economics or management (En);

Ranking following from these observations

$$a \succ_O b \succ_O c \succ_O d \succ_O e \succ_O f \succ_O g,$$

There is **no additive model** that can lead to this ranking.

Capacities

Let $\mathcal{P}(N)$ denote the power set of $N := \{1, \ldots, n\}$.

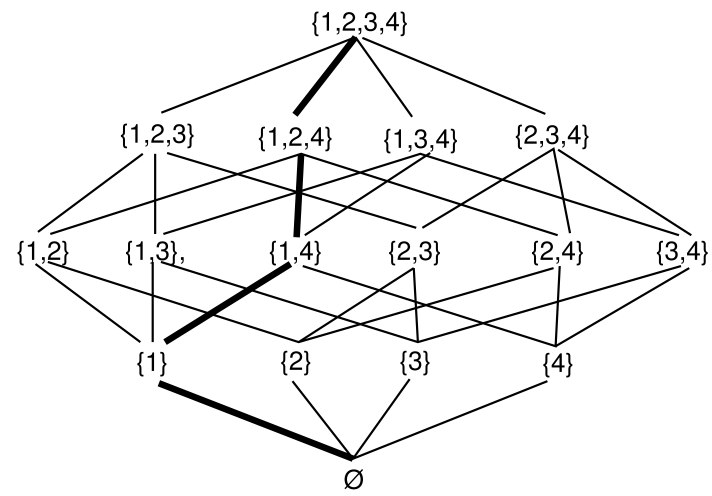
Definition

A function $\mu : \mathcal{P}(N) \to [0,1]$ is a **capacity** if it satisfies :

- $\mu(\emptyset) = 0$, $\mu(N) = 1$,
- for any $S, T \subseteq N$, $S \subseteq T \Rightarrow \mu(S) \leq \mu(T)$.

In the context of aggregation, the **importance** of the subsets of (**interacting**) criteria can be modeled by a capacity.

Capacities



The Choquet integral

The **Choquet integral** can be regarded as a natural extension of the weighted arithmetic mean.

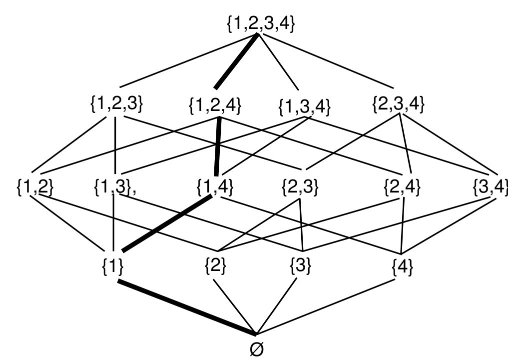
Definition

The Choquet integral of $x = (x_1, \ldots, x_n)$ **w.r.t a capacity** μ on N is defined by

$$C_\mu(x) := \sum_{i=1}^{n} x_{\sigma(i)}[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i+1)})],$$

where σ is a permutation on N such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$, $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$, for all $i \in \{1, \ldots, n\}$, and $A_{\sigma(n+1)} := \emptyset$.

The Choquet integral



Example : $n = 4$. If $x_3 \leq x_2 \leq x_4 \leq x_1$, we have

$$C_\mu(x_1, x_2, x_3, x_4) = x_3\,[\mu(3,2,4,1) - \mu(2,4,1)] + x_2\,[\mu(2,4,1) - \mu(4,1)] + x_4\,[\mu(4,1) - \mu(1)] + x_1\,[\mu(1) - \mu(\emptyset)].$$

The identification problem

Seen as an aggregation operator, the Choquet integral can be regarded as defined by $2^n - 2$ coefficients, which allows a **greater flexibility**.

Nota bene : in practise, special classes of Choquet integrals defined by less parameters are usually used ; they are obtained by considering **k-additive capacities**.

Drawback: in any case, the decision maker will have difficulties to provide all the parameters.

The identification problem : How to determine the underlying capacity μ?

The identification problem

In practise : only a finite and usually small subset $\mathcal{O}$ of the set X of objects of interest is available.

General form of the initial preference of the DM:

- a ranking of the available objects;
- a ranking of the importance of the criteria;
- quantitative intuitions about the importance of some criteria;
- a ranking of interactions between criteria;
- intuitions about the type and the magnitude of the interaction between some criteria;
- the behaviour of some criteria as **veto** or **favour**;
- etc.

Existing approaches

Least squares based approaches (Mori and Murofushi 1989)
Minimise the average quadratic distance between overall utilities

An approach based on linear programming (Marichal and Roubens 2000)
Maximise the minimal difference between the overall utilities of objects in a given ranking

Minimum variance and minimum distance approaches (Kojadinovic 2006)
Favour the "least specific" capacity compatible with the initial preferences of the DM

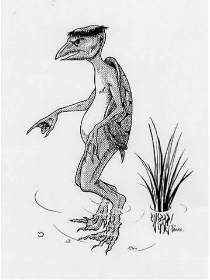
A less constrained approach (Meyer and Roubens 2005)
Can provide a solution even if some initial preferences of the DM are not compatible with a Choquet integral model

The Kappalab R toolbox

Most of the existing approaches have been implemented within the **Kappalab R package**

Kappalab, which stands for "laboratory for capacities", is a package for the GNU R statistical system

GNU R is a (Matlab like) free software environment for statistical computing and graphics. It compiles and runs on a wide variety of UNIX platforms, Windows and Mac OS



Back to the example

Define seven vectors representing the profiles of the students :

```
> a <- c(18,11,11,11,18)
> b <- c(18,11,18,11,11)
> c <- c(11,11,18,11,18)
> d <- c(18,18,11,11,11)
> e <- c(11,11,18,18,11)
> f <- c(11,11,18,11,11)
> g <- c(11,11,11,11,18)
```

Store the preference threshold on the overall utilities in an R variable :

```
> delta.C <- 0.5
```


Back to the example

The **weak order** over the students is encoded into a 6 row R matrix :

```
> Acp <- rbind(c(a,b,delta.C),
               c(b,c,delta.C),
               c(c,d,delta.C),
               c(d,e,delta.C),
               c(e,f,delta.C),
               c(f,g,delta.C))
```

each row containing a constraint of the form $C_{m_\nu}(u(x)) \geq C_{m_\nu}(u(y)) + \delta_C$.

For example, the **minimum variance** approach is invoked by typing (search for a 2-additive capacity) :

```
> mv <- mini.var.capa.ident(5,2,A.Choquet.preorder = Acp)
```


Back to the example

The **solution**, a 2-additive capacity given under the form of its Möbius representation, can be obtained by typing:

```
> m <- mv$solution
```

and visualised by entering `m` in the R terminal:

```
> m
       Mobius.capacity
{}     0.00
{1}    0.46
{2}    0.30
...    ...
{3,5}  -0.06
{4,5}  -0.07
```


Back to the example

The **overall utilities** computed using the Choquet integral w.r.t. the 2-additive solutions (for the LP, the MV and the MD methods) :

```
  S  P  E  M En Mean    LP    MV    MD
a 18 11 11 11 18 13.8 18.00 15.25 14.95
b 18 11 18 11 11 13.8 17.36 14.75 14.45
c 11 11 18 11 18 13.8 16.73 14.25 13.95
d 18 18 11 11 11 13.8 16.09 13.75 13.45
e 11 11 18 18 11 13.8 15.45 13.25 12.95
f 11 11 18 11 11 12.4 14.82 12.75 12.45
g 11 11 11 11 18 12.4 14.18 12.25 11.95
```

(in accordance with the DM's preferences on the ranking of the 7 students)

Back to the example

The Shapley values of the 2-additive solutions are:

	S	P	E	M	En
LP	0.45	0.00	0.27	0.05	0.23
MV	0.27	0.16	0.21	0.14	0.22
MD	0.24	0.18	0.20	0.16	0.22

Statistics is the most important criterion

The overall importances of the criteria are **not in accordance** with the orientation of the institution ($S \sim P$ and $E \sim M$)

Can be **fixed** by imposing additional constraints

Back to the example

Additional constraints on the Shapley value

The DM explicitly requires that statistics (S) and probability (P), and economics (E) and management (M), have the same overall importances :

$-0.1 \leq \phi_{m_\nu}(S) - \phi_{m_\nu}(P) \leq 0.1$ and $-0.1 \leq \phi_{m_\nu}(E) - \phi_{m_\nu}(M) \leq 0.1$

These inequalities are encoded into a 4 row R matrix:

```
> Asp <- rbind(c(1,2,-0.1),
               c(2,1,-0.1),
               c(3,4,-0.1),
               c(4,3,-0.1))
```

The minimum variance approach invoked by entering:

```
> mv2 <- mini.var.capa.ident(5,2,A.Choquet.preorder = Acp,
                             A.Shapley.preorder = Asp)
```


Back to the example

The **Shapley values** of the 2-additive solutions are:

	S	P	E	M	En
LP	0.23	0.23	0.18	0.18	0.18
MV	0.22	0.21	0.18	0.17	0.22
MD	0.22	0.21	0.18	0.17	0.22

As expected, the solutions satisfy the constraints additionally imposed by the DM.