The RuBy method for the recommendation of a best choice from a bipolar outranking relation

Patrick Meyer

University of Luxembourg

Joint work with Raymond Bisdorff and Marc Roubens
27th Linz Seminar on Fuzzy Set Theory

sma Service de Mathématiques Appliquées

UNIVERSITÉ DU LUXEMBOURG

Context

Multi-criteria decision support

Alternatives $\{x, y, z, \ldots\}$ are described by a set of criteria

"*At least as good as*" preferences are modelled by a bipolar valued outranking relation S

Definition [Roy 1985, Roy Bouyssou 1993]

An outranking xSy holds if there is a sufficient majority of criteria which is in favour of this assertion and there is absence of veto on all criteria

The concordance produces a valuation of the outranking relation

This valuation expresses the likelyhood of a global pairwise preference situation

This valuation is bipolar, by construction

The outranking relation is generally neither complete nor transitive

The bipolar valued outranking relation

Let $\widetilde{\mathrm{S}} : X \times X \rightarrow \mathcal{L}$ be the bipolar valued characterisation of the outranking relation S on X

$\mathcal{L} = \{-m, \ldots, 0, \ldots, +m\}$ is a bipolar evaluation domain

The values of $\mathcal{L}$ express a degree of likelihood

Interpretation

$\widetilde{\mathrm{S}}(x,y) = +m$: it is certainly true that x outranks y
$\widetilde{\mathrm{S}}(x,y) > 0$: it is more true than false that x outranks y
$\widetilde{\mathrm{S}}(x,y) = 0$: it is neither true nor false that x outranks y
$\widetilde{\mathrm{S}}(x,y) < 0$: it is more false than true that x outranks y
$\widetilde{\mathrm{S}}(x,y) = -m$: it is certainly false that x outranks y

$\widetilde{\mathrm{S}}$ combined to X defines a bipolar valued outranking digraph $\widetilde{G}^{\mathcal{L}}$

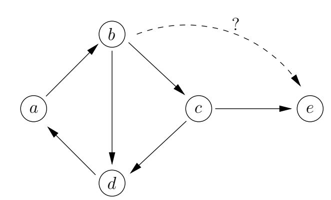
Remark: the valuation does not express an evaluation of a difference between alternatives

The best choice problem (BCP)

Goal

To build a best choice recommendation from a bipolar valued outranking relation on a finite set X of decision alternatives

$\widetilde{\mathrm{S}}$	a	b	c	d	e
a	-	2	-10	-4	-8
b	-6	-	8	10	0
c	-10	-10	-	2	8
d	6	-6	-10	-	-4
e	-10	-8	-4	-6	-



Which alternative is the best choice ?

Outline of the presentation

Preliminaries

5 pragmatic principles

Their translation in the outranking digraph

The RuBy best choice recommendation

Application

Preliminaries

Search for a single best choice (*1-best* choice)[1]

Progressive method :

- caution and economical reason
- $\longrightarrow$ possible iterations and re-questionings of the DM

Inspired from Roy and Bouyssou's seminal work on the Electre I and Electre IS methods

Starting point : inadequacy of existing approaches

Based on five pragmatic principles

[1] The k-BCP can be solved like a 1-BCP by a proper modification of X

Preliminaries

Definition

A choice Y is a subset of alternatives of X

Definition

A best choice Y is a subset of alternatives of X which must help the DM to focus on the next step of the progressive search for a best alternative

The RuBy principles

A best choice Y should verify the five following pragmatic principles:

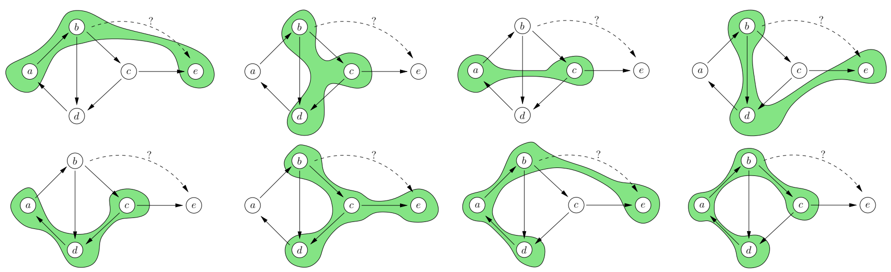
- $\mathcal{B}_1$ Each not retained alternative must be rejected for well motivated reasons
- $\mathcal{B}_2$ The number of retained alternatives should be as small as possible
- $\mathcal{B}_3$ The subset Y of retained alternatives should not be simultaneously a "best" and a "worst" choice (effectiveness)
- $\mathcal{B}_4$ Each step of the progressive search must be an efficient and informative refinement of the previous recommendation
- $\mathcal{B}_5$ The best choice Y must be robust (with respect to impreciseness in the data) (robustness)

$\mathcal{B}_1$: rejection for well motivated reasons

Translation in $\widetilde{G}^{\mathcal{L}}$

Every not retained alternative must be outranked by at least one alternative of the best choice; i.e. the best choice Y must be an outranking choice

Example :



The RuBy principles

A best choice Y should verify the five following pragmatic principles:

- $\mathcal{B}_1$ Each not retained alternative must be rejected for well motivated reasons
- $\mathcal{B}_2$ The number of retained alternatives should be as small as possible
- $\mathcal{B}_3$ The subset Y of retained alternatives should not be simultaneously a "best" and a "worst" choice (effectiveness)
- $\mathcal{B}_4$ Each step of the progressive search must be an efficient and informative refinement of the previous recommendation
- $\mathcal{B}_5$ The best choice Y must be robust (with respect to impreciseness in the data) (robustness)

$\mathcal{B}_2$: minimal size

Definition

The outranking neighbourhood $\Gamma^+(x)$ of an alternative x is the union of x and the set of alternatives which are outranked by x

Definition

The outranking private neighbourhood $\Gamma^+_Y(x)$ of an alternative x in a choice Y is the set $\Gamma^+(x) \setminus \Gamma^+(Y \setminus \{x\})$

Definition

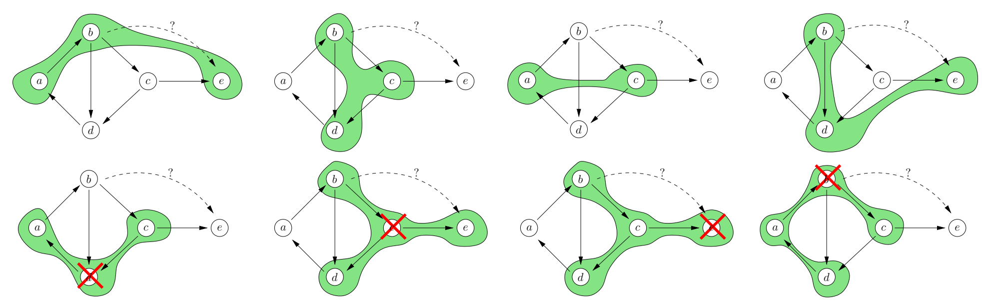
A choice Y is said to be irredundant if all the nodes of Y have non-empty private neighbourhoods

Translation of the minimality in $\widetilde{G}^{\mathcal{L}}$

The subset Y of retained alternatives must be irredundant

$\mathcal{B}_2$: minimal size

Example :



Note 1: $\mathcal{B}_1$ and $\mathcal{B}_2$ were already introduced in [Roy 1981, Roy Bouyssou 1993] in methodological studies on the choice problem
→ Use of the outranking kernel of an outranking digraph as the best choice

Note 2: Principles $\mathcal{B}_1$ and $\mathcal{B}_2$ produce the minimal outranking choices of the outranking digraph

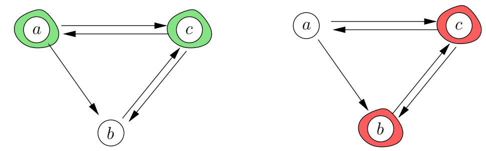
The RuBy principles

A best choice Y should verify the five following pragmatic principles:

- $\mathcal{B}_1$ Each not retained alternative must be rejected for well motivated reasons
- $\mathcal{B}_2$ The number of retained alternatives should be as small as possible
- $\mathcal{B}_3$ The subset Y of retained alternatives should not be simultaneously a "best" and a "worst" choice (effectiveness)
- $\mathcal{B}_4$ Each step of the progressive search must be an efficient and informative refinement of the previous recommendation
- $\mathcal{B}_5$ The best choice Y must be robust (with respect to impreciseness in the data) (robustness)

$\mathcal{B}_3$: effectiveness of the outranking

Example :



$\{a\}$ and $\{c\}$ both verify $\mathcal{B}_1$ and $\mathcal{B}_2$, are both optimal, but not equivalent

$\{c\}$ is conjointly a best and a worst choice

Translation in $\widetilde{G}^{\mathcal{L}}$

The best choice Y must be a strict best choice, i.e. it must not be conjointly an outranked and an outranking choice (at the same level of determination)

The RuBy principles

A best choice Y should verify the five following pragmatic principles:

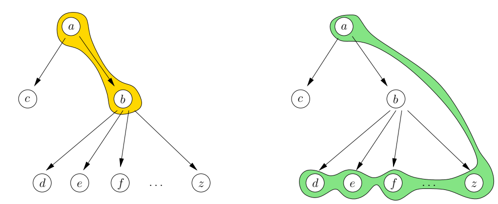
- $\mathcal{B}_1$ Each not retained alternative must be rejected for well motivated reasons
- $\mathcal{B}_2$ The number of retained alternatives should be as small as possible
- $\mathcal{B}_3$ The subset Y of retained alternatives should not be simultaneously a "best" and a "worst" choice (effectiveness)
- $\mathcal{B}_4$ Each step of the progressive search must be an efficient and informative refinement of the previous recommendation
- $\mathcal{B}_5$ The best choice Y must be robust (with respect to impreciseness in the data) (robustness)

$\mathcal{B}_4$: efficient refinement

Motivation: The next step of the search for the best choice is oriented towards a set of alternatives which is interesting for a further analysis

Consequence: Y cannot contain a *further* best choice

Example :



Observations :

Very empty graph !

Why are d, e, ..., z not outranked by a ?

In the next step of the progressive search these questions may be answered

$\mathcal{B}_4$: efficient refinement

Translation in $\widetilde{G}^{\mathcal{L}}$

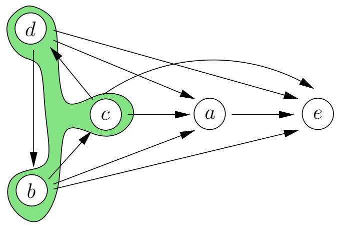
A best choice Y is stable[a] iff it is composed of unconnected cordless circuits of odd order[b]

[a] The subgraph restricted to the nodes of Y does not admit any further best choice

[b] A singleton is considered as a circuit of order 1

Justification : If a graph contains no circuit of odd order then it has an outranking kernel (Richardson)

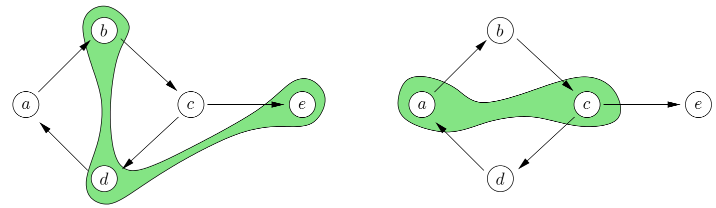
Example :



$\mathcal{B}_4$: efficient refinement

Remark: If the outranking digraph contains no circuits of odd order (> 1), then $\mathcal{B}_1$ & $\mathcal{B}_2$ & $\mathcal{B}_4$ produce the outranking kernels of the digraph as best choices

Example :



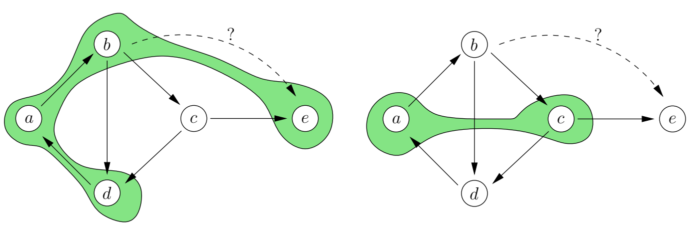
$\mathcal{B}_4$: efficient refinement

Hyper-kernels

A choice Y which is composed of unconnected cordless circuits of odd order (≥ 1) is called a *hyper-kernel*

In a general outranking digraph the principles $\mathcal{B}_1$ & $\mathcal{B}_2$ & $\mathcal{B}_4$ produce the outranking hyper-kernels of the digraph as best choices

Example :



The RuBy principles

A best choice Y should verify the five following pragmatic principles:

- $\mathcal{B}_1$ Each not retained alternative must be rejected for well motivated reasons
- $\mathcal{B}_2$ The number of retained alternatives should be as small as possible
- $\mathcal{B}_3$ The subset Y of retained alternatives should not be simultaneously a "best" and a "worst" choice (effectiveness)
- $\mathcal{B}_4$ Each step of the progressive search must be an efficient and informative refinement of the previous recommendation
- $\mathcal{B}_5$ The best choice Y must be robust (with respect to impreciseness in the data) (robustness)

$\mathcal{B}_5$: robustness

Preliminaries :

A choice Y can be characterised by a membership vector

Formally : $[\widetilde{Y}(x), x \in X]$

$\widetilde{Y}(x)$ is the degree of credibility of the assertion "*alternative x belongs to the choice Y*"

The outranking / outranked kernels of the outranking digraph are among the solutions of the following equation systems [Bisdorff Roubens 1996]:

$$(\widetilde{Y} \circ \widetilde{S})(x) = \max_{y \neq x}[\min(\widetilde{Y}(y), \widetilde{S}(y,x))] = -\widetilde{Y}(x) \quad (1)$$

$$(\widetilde{S} \circ \widetilde{Y}^t)(x) = \max_{y \neq x}[\min(\widetilde{S}(x,y), \widetilde{Y}(y))] = -\widetilde{Y}(x) \quad (2)$$

$\mathcal{B}_5$: robustness

General observations :

The evaluations of the alternatives on the criteria may be imprecise or even missing

The indifference and preference threshold are not known with absolute preciseness

The estimated importance of the criteria may suffer from impreciseness

...

Requirement

The output of the best choice procedure should not depend on impreciseness in the data

$\mathcal{B}_5$: robustness

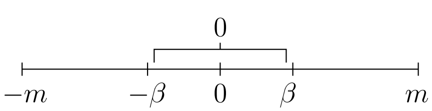
Question:
How should the bipolar valued outranking relation be tuned to deal with this robustness issue ?

Answer:
The median valuation 0 ("*it is neither true nor false that x outranks y*") must be extended to a larger interval of values

β-cut

The symmetrical *beta*-cut $\widetilde{S}_\beta$ of $\widetilde{S}$ is defined as follows ($\beta \in]0, m[$):

$$\widetilde{S}_\beta(x,y) = \begin{cases} \widetilde{S}(x,y) & \text{if } |\widetilde{S}(x,y)| \geq \beta \\ 0 & \text{else} \end{cases}$$



$\mathcal{B}_5$: robustness

The symmetrical β-cut allows to extend the zone of indetermination

$\Rightarrow$ a larger set of arrows in the outranking digraph will be considered as indetermined

"*Unanimity* remains *unanimity*"

Translation in $\widetilde{G}^{\mathcal{L}}$

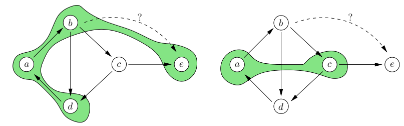
The best choice Y must withstand the highest possible progressive symmetrical β-cuts of the outranking relation $\widetilde{S}$

$\mathcal{B}_5$: robustness

Example :

$\widetilde{S}$:

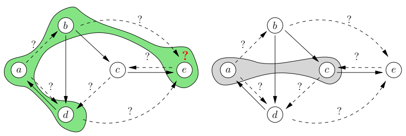
$\widetilde{S}$	a	b	c	d	e
a	-	2	-10	-4	-8
b	-6	-	8	10	0
c	-10	-10	-	2	8
d	6	-6	-10	-	-4
e	-10	-8	-4	-6	-



choice	$\{a, b, d\}$	a	b	c	d	e
$\{\{a, b, d\}, e\}^+$	6	-6	-6	-6	-6	0
$\{a, c\}^+$	-2	2	-2	2	-2	-2

$\widetilde{S}_{\beta \geq 6}$:

$\widetilde{S}$	a	b	c	d	e
a	-	0	-10	0	-8
b	-6	-	8	10	0
c	-10	-10	-	0	8
d	6	-6	-10	-	0
e	-10	-8	0	-6	-



choice	$\{a, b, d\}$	a	b	c	d	e
$\{\{a, b, d\}, e\}^+$	6	-6	-6	-6	-6	0
$\{a, c\}^+$	0	0	0	0	0	0

The COCA outranking digraph

Recall: A choice Y which is composed of unconnected cordless circuits of odd order (≥ 1) is called a hyper-kernel

According to $\mathcal{B}_4$ the best choices are among the hyper-kernels of the outranking digraph

The cordless circuits of odd order are added to $\widetilde{G}^{\mathcal{L}}$ as new nodes

The original *problematic* circuits of odd order are "hidden" behind these new nodes

The COCA outranking digraph

Construction of the **C**ordless-**O**dd-**C**ircuits-**A**ugmented outranking digraph:

Initialisation : $\widetilde{G}^{\mathcal{L}}_0(X_0, \widetilde{S}_0) := \widetilde{G}^{\mathcal{L}}(X, \widetilde{S})$

Step i : $X_i = X_{i-1} \cup \mathcal{C}_i$, where $\mathcal{C}_i$ is a set of nodes representing the cordless circuits of odd order of $\widetilde{G}^{\mathcal{L}}_{i-1}(X_{i-1}, \widetilde{S}_{i-1})$

$$\forall C_k \in \mathcal{C}_i \begin{cases} \widetilde{S}_i(C_k, x) = \bigcup_{y \in C_k} \widetilde{S}_{i-1}(y, x) & \forall x \in X_{i-1} \setminus C_k \\ \widetilde{S}_i(C_k, x) = m & \forall x \in C_k \end{cases},$$

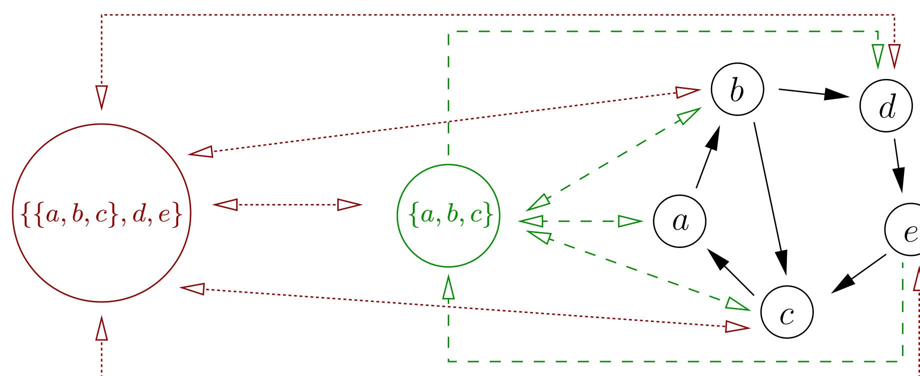
$$\forall x \in X_{i-1}, C_k \in \mathcal{C}_i \begin{cases} \widetilde{S}_i(x, C_k) = \bigcup_{y \in C_k} \widetilde{S}_{i-1}(x, y) & \text{if } x \notin C_k \\ \widetilde{S}_i(x, C_k) = m & \text{if } x \in C_k \end{cases}$$

Stop : at step r when $|X_r| = |X_{r+1}|$

The COCA outranking digraph

Iterative approach because new odd circuits may appear !

Example :



The COCA outranking digraph

The hyper-kernels of the outranking digraph $\widetilde{G}^{\mathcal{L}}$ are the kernels of the COCA outranking digraph

Property:
The outranking kernels of $\widetilde{G}^{\mathcal{L}}$ are among the outranking kernels of the COCA outranking digraph

Property:
The COCA outranking digraph contains at least one outranking hyper-kernel

Corollary

The RuBy method always produces an answer

The COCA outranking digraph

Remark : hyper-kernels of the COCA outranking digraph which are represented by characteristic vectors containing some median values are called pre-hyper-kernels

Theorem

A best choice in a given outranking graph $\widetilde{G}^{\mathcal{L}}(X, \widetilde{S})$ verifies all five RuBy principles if and only if it is a strict logically most determined outranking (pre-)hyper-kernel of the associated COCA outranking digraph

Algorithm

Input : $\widetilde{G}^{\mathcal{L}}(X, \widetilde{S})$

1. Construction of the associated COCA outranking digraph (detection of the cordless odd circuits)
2. Extraction of all strict outranking (pre-)hyper-kernels of the COCA outranking digraph
3. Sorting of the (pre-)hyper-kernels by descending logical determination

Output : The RuBy best choice recommendation is given by the strict logically most determined outranking (pre-)hyper-kernel

Which is the best German university ?

Online survey "Studentenspiegel April-Juni 2004" (Der Spiegel 48/2004 p.181)

X = 40 German universities (München, Freiburg, Heidelberg, Mannheim, Tübingen, ...)

15 disciplines (management sciences, biology, modern literature, law, engineering, ...)

3 categories per discipline (high quality, average quality, low quality) according to the professional insertion of the alumni

Which is the best German university ?

15 qualitative criteria using a 3-levelled ordinal scale (0 = low quality, 1 = average quality, 2 = superior quality)

We assume that the 15 disciplines are equi-significant

Ordinal scale on each criterion:

- indifference threshold : 0
- preference threshold : 1
- veto threshold : 3

There are missing evaluations

Which is the best German university ?

The bipolar valued outranking digraph presents four cordless circuits of order 3 :

1. {Berlin Humboldt, Kaiserslautern, Mannheim}
2. {Berlin Humboldt, Kaiserslautern, Marburg}
3. {Chemnitz, Kaiserslautern, Marburg}
4. {Chemnitz, Freie Universität Berlin, Kaiserslautern}

Which is the best German university ?

RuBy best choices in decreasing order of their logical determination (valuation $[0, 100]$):

1. Universtät München at level 65%
2. Universtät Freiburg, at level 63%
2. Universtät Leipzig, at level 63%
4. Universtät Augsburg, at level 55%
4. {Berlin Humboldt, Kaiserslautern, Mannheim} at level 55%
6. Technische Universtät München, at level 55%
7. Universtät Konstanz, at level 53%

The Universtät München is the RuBy best choice recommendation in terms of the quality of the professional insertion of its alumni

Concluding remarks

1-best choice problem

Progressive method

Based on five pragmatic principles

No modification of the original problem

Always a best choice recommendation.