On the use of the Choquet integral with fuzzy numbers in Multiple Criteria Decision Aiding

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A general case

The evaluations of the alternatives on the points of view can suffer from impreciseness

Example:

\( A = \) set of alternatives

\( N = \) set of points of view

\( A = \) set of houses

\( N = \) \{price, surface area, age, \ldots\}

Each alternative is evaluated on each point of view

Objective

Build a ranking on the alternatives

Fuzzy sets

Zadeh (1965) introduces the concept of fuzzy sets to represent impreciseness

Definition

A fuzzy subset \( \tilde{Y} \) of a classical set \( X \) is defined by its membership function

\[ \mu_{\tilde{Y}} : X \to [0, 1]. \]

For \( x \in X \):

\[ \mu_{\tilde{Y}}(x) = 0 : x \text{ does not belong to } \tilde{Y} \]
\[ \mu_{\tilde{Y}}(x) = 1 : \text{complete membership of } x \text{ to } \tilde{Y} \]
\[ \mu_{\tilde{Y}}(x) \in [0, 1[ : \text{intermediate memberships} \]

Notation: \( \tilde{Y}(x) := \mu_{\tilde{Y}}(x) \)
**Fuzzy numbers**

**Definition**

A fuzzy number $\tilde{x}$ of $\mathbb{R}$ is a fuzzy subset of $\mathbb{R}$ that is normal\(^a\), fuzzy convex\(^b\) and has a continuous membership function of bounded support.

\(^a\) There exists $x \in \tilde{x}$ for which $\tilde{x}(x) = 1$

\(^b\) Any $\lambda$-cut is convex

**General way to write a fuzzy number**

For $\tilde{x} \in \mathcal{F}$:

$$a_m(\lambda) = \min[\tilde{x}]^\lambda, \quad a_M(\lambda) = \max[\tilde{x}]^\lambda$$

for each $\lambda \in [0, 1].$

where $[a, b]$ is the peak of $\tilde{x},$

$L, R : [0, 1] \to [0, 1]$ are upper semi-continuous and non-increasing functions, $L(0) = R(0) = 1$ and $L(1) = R(1) = 0.$

$L, R$ are called side functions.

The support $\tilde{x}$ is equal to $]a - \alpha, b + \beta[.$
Trapezoidal fuzzy numbers

Very useful in Multiple Criteria Decision Aiding

\[ x(x) = \begin{cases} 
1 - \frac{x - a}{\alpha} & \text{if } a - \alpha \leq x \leq a, \\
1 & \text{if } a < x \leq b, \\
1 - \frac{x - b}{\beta} & \text{if } b < x \leq b + \beta, \\
0 & \text{otherwise.}
\end{cases} \]

\( \alpha \) and \( \beta \) are called the left and right width,
notation \( \tilde{x} = (a, b, \alpha, \beta) \).

If \( a = b \) the fuzzy number is called triangular and \( a \) is said to be the center of \( \tilde{x} \).

Possibility distributions and fuzzy numbers

Let \( a, b \in \mathbb{R} \cup \{-\infty, +\infty\} \) with \( a \leq b \)

**Definition**

The possibility that \( \tilde{x} \in F \) takes its value from the interval \([a, b]\) is defined by

\[
\text{Pos}(\tilde{x} \in [a, b]) = \max_{x \in [a, b]} x(x).
\]

In particular for \( \lambda \in [0, 1] \),

\[
\text{Pos}(\tilde{x} \leq a_m(\lambda)) = \max_{x \leq a_m(\lambda)} x(x) = \lambda,
\]

\[
\text{Pos}(\tilde{x} \geq a_M(\lambda)) = \max_{x \geq a_M(\lambda)} x(x) = \lambda.
\]

Zadeh’s extension principle

**Goal:** To use mathematical operations on the fuzzy numbers, compatible with classical arithmetics

If \( \tilde{x}_1, \ldots, \tilde{x}_n \in F \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuous function, then the sup-min extension principle of Zadeh is defined by

\[
f(\tilde{x}_1, \ldots, \tilde{x}_n)(y) = \sup_{f(x_1, \ldots, x_n) = y} \min\{\tilde{x}_1(x_1), \ldots, \tilde{x}_n(x_n)\}, \forall y \in \mathbb{R}.
\]

It has been shown that \( f(\tilde{x}_1, \ldots, \tilde{x}_n) \) is a fuzzy number.

Preliminary considerations

\( A = \) set of \( q \) alternatives

\( N = \{1, \ldots, n\} \) set of labels of points of view

Partial evaluations of each alternative on each point of view made on an interval scale

\( \Rightarrow \) admissible transformation function: positive linear transformation

Evaluations supposed to be commensurable

Performance scale of each point of view: \([0, 1]\)
Considerations on fuzzy numbers

The MCDA framework

Ranking the alternatives

The assessment of the parameters

Preliminary considerations

Work hypothesis

Choquet: The classical case

Choquet: A fuzzy extension

Objective

Rank the alternatives

1. Aggregate the partial evaluations of each alternative on the points of view in a global evaluation,
2. Order these global evaluations to obtain a ranking.

In general

Use of the weighted sum as an aggregator.

Disadvantage

Supposes preferential independence of the points of view

One possible solution

Use of the Choquet integral as an aggregator

Work hypothesis: preliminaries

Let $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathcal{F}$

Definition

$c \in \mathbb{R}^n$ is called the joint possibility distribution of $\tilde{x}_1, \ldots, \tilde{x}_n$ if it satisfies

$$c(x_1, \ldots, x_n) \leq \min\{\tilde{x}_1(x_1), \ldots, \tilde{x}_n(x_n)\} \quad \forall x_i \in \mathbb{R}, i = 1, \ldots, n.$$ 

Equivalently: $[c]^\gamma \subseteq [\tilde{x}_1]^\gamma \times \ldots \times [\tilde{x}_n]^\gamma \forall \gamma \in [0, 1]$

Fuzzy numbers $\tilde{x}_i \in \mathcal{F}$, $i = 1, \ldots, n$ are said to be non-interactive if

$[c]^\gamma = [\tilde{x}_1]^\gamma \times \ldots \times [\tilde{x}_n]^\gamma$ for all $\gamma \in [0, 1]$

Zadeh implicitly supposes that the fuzzy numbers have to be non-interactive in the extension principle.

Figure: Non-interactive (left) and interactive (right) fuzzy numbers
Work hypothesis

The partial evaluations of the alternatives are non-interactive (in the sense discussed before)

But:

Points of view can interact

- substitutiveness
- complementarity
- preferential dependence

The satisfaction of one point of view has almost the same effect as the satisfaction of both.

The satisfaction of only one point of view is weak compared to the satisfaction of both points of view.

Consider \( x, y \in A \) for which the evaluations on \( S \subseteq N \) are equal. The subset \( N \setminus S \) is preferentially independent of \( S \) if the preference of \( x \) over \( y \) is not influenced by their common part on \( S \).

Advantages and drawback

Allows to represent interactions

Large number of parameters \( (2^n - 2) \Rightarrow \) great flexibility

But: difficult to assign values to the parameters of \( v \)

Towards an equivalent representation

\( v \) fuzzy measure on \( N \)

The Möbius transform of \( v \) is a set function \( m : 2^N \rightarrow \mathbb{R} \) defined by

\[
m(S) = \sum_{T \subseteq S} (-1)^{|S| - |T|} v(T) \quad (S \subseteq N)
\]

invertible transformation

\( v \) can be recovered from \( m \)

The Choquet integral

\( x \in A \), described by crisp partial evaluations \( x_i, (i \in N) \)

Definition

The Choquet integral of \( x \) is defined by:

\[
C_v(x) := \sum_{i=1}^{n} x_i [v(A_i) - v(A_{i+1})]
\]

where \( v \) represents a fuzzy measure on \( N \), that is a monotone set function \( v : 2^N \rightarrow [0, 1] \) fulfilling \( v(\emptyset) = 0 \) and \( v(N) = 1 \).

The parentheses used for indexes represent a permutation on \( N \) such that

\( x_1 \leq \ldots \leq x_n \),

and \( A(i) \) represents the subset \( \{(i), \ldots , (n)\} \).

Considerations on fuzzy numbers

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Preliminary considerations
Work hypothesis
Choquet: The classical case
Choquet: A fuzzy extension
Choquet integral with $m$

$m$ set function on $N$

**Definition**

$$C_m(x) = \sum_{T \subseteq N} m(T) \bigwedge_{i \in T} x_i$$

Main difference: no reordering of the partial evaluations $x_i$

**$k$-additivity**

A fuzzy measure is $k$-additive if its Möbius transform $m$ satisfies $m(S) = 0$ for $S$ such that $|S| > k$ and there exists at least one subset $S$ such that $|S| = k$ and $m(S) \neq \emptyset$.

Towards a fuzzy extension of the Choquet integral

Definition of the Choquet integral in terms of a set function $m$ is a combination of functions which are continuous on $\mathbb{R} \times \mathbb{R}$: addition ($+$), multiplication ($\cdot$) and minimum ($\wedge$)

Extension principle of Zadeh: extend the three functions to their fuzzy versions

$p \in \mathbb{R}$ (a coefficient of $m$), $\tilde{x}_1 \in \mathcal{F}$

$$[p \cdot \tilde{x}_1]^\lambda = p[\tilde{x}_1]^\lambda$$

for any $\lambda \in [0, 1]$

Result: fuzzy number

Trapezoidal fuzzy numbers

$$\tilde{x} = (a_x, b_x, \alpha_x, \beta_x) \text{ and } \tilde{y} = (a_y, b_y, \alpha_y, \beta_y)$$

$$\tilde{x} + \tilde{y} = (a_x + a_y, b_x + b_y, \alpha_x + \alpha_y, \beta_x + \beta_y)$$

Result: fuzzy number

Trapezoidal fuzzy numbers

$$\rho \tilde{x} = (\rho a_x, \rho b_x, \rho \alpha_x, \rho \beta_x)$$

$$p\tilde{x} = (p a_x, p b_x, p \alpha_x, p \beta_x).$$
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A fuzzy extension of the Choquet integral

Definition
\[ \tilde{C}_m(x) = \sum_{T \subseteq \mathbb{N}} m(T) \cdot \bigwedge_{i \in T} x_i. \]

where \( m \) is the set function which is obtained by a Möbius transform of the fuzzy measure \( \nu \).

Result: fuzzy number
Trapezoidal fuzzy numbers
side functions become piecewise linear functions

Example

\( N = \{1, 2\} \), \( \nu(\{1\}) = 0.1 \), \( \nu(\{2\}) = 0.2 \), \( \nu(\{1, 2\}) = 1 \)
\( m(\{1\}) = 0.1 \), \( m(\{2\}) = 0.2 \), \( m(\{1, 2\}) = 0.7 \)
\( \tilde{x}_1 = (1, 1, 0) \) and \( \tilde{x}_2 = (0.5, 0, 1) \)
\( \tilde{C}_m((\tilde{x}_1, \tilde{x}_2)) = m(\{1\}) \cdot \bigwedge \tilde{x}_1 + m(\{2\}) \cdot \bigwedge \tilde{x}_2 + m(\{1, 2\}) \cdot \bigwedge (\tilde{x}_1, \tilde{x}_2) \)
\[ = (0.2, 0.1, 0.2) \cdot 0.7 \cdot \bigwedge (\tilde{x}_1, \tilde{x}_2) \]
\[ := \tilde{s} + \tilde{s}' \]

where \( \tilde{s} = (0.2, 0.1, 0.2) \) and \( \tilde{s}' = 0.7 \cdot \bigwedge (\tilde{x}_1, \tilde{x}_2) \).

(1) and (2) stand respectively for \( \tilde{x}_1 \) and \( \tilde{x}_2 \)
(3) and (4) represent \( \tilde{s} \) and \( \tilde{s}' \)
(5) shows the aggregated value \( \tilde{C}_m((\tilde{x}_1, \tilde{x}_2)) \)

Figure: Example: towards the aggregation
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Calculatory considerations

Possibilistic mean

Consider the membership function of a fuzzy number \( \tilde{x} \) as a possibility distribution

Upper possibilistic mean of \( \tilde{x} \)

\[
\mathcal{M}^+(\tilde{x}) := \int_0^1 a_M(\lambda) d\lambda = \int_0^1 a_M(\lambda) d\text{Pos}(\tilde{x} \geq a_M(\lambda))
\]

Possibilistic mean of \( \tilde{x} \)

\[
\mathcal{M}(\tilde{x}) := \frac{1}{2}[\mathcal{M}^+(\tilde{x}) + \mathcal{M}^-(\tilde{x})].
\]

Lower possibilistic mean of \( \tilde{x} \)

\[
\mathcal{M}^-(\tilde{x}) := \int_0^1 a_m(\lambda) d\lambda = \int_0^1 a_m(\lambda) d\text{Pos}(\tilde{x} \leq a_m(\lambda)).
\]

Ranking by a complete preorder

Ranking by comparing the possibilistic means of the global evaluations of the alternatives of \( A \)

\[
x, y \in A. \text{ Complete preorder } \preceq (\text{is not worse than}): \]

\[
x \preceq y \iff \mathcal{M}(\tilde{C}_m(x)) \geq \mathcal{M}(\tilde{C}_m(y))
\]

Geometric interpretation

It is possible to show that

\[
x \preceq y \iff A_4 + A_1 \geq A_3 + A_2.
\]

Corresponds to the area compensation method of Fortemps and Roubens (1996).
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By a complete preorder
By a complete interval order
By a complete fuzzy interval order
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Ranking by a complete interval order

Ranking by comparing the intervals \([\mathcal{M}^- (\tilde{C}_m(x)), \mathcal{M}^+(\tilde{C}_m(x))]\)

\(I\) symmetrical relation, \(P\) antisymmetrical relation on \(A \times A\) such that

\[
\begin{align*}
  xPy & \iff \mathcal{M}^- (\tilde{C}_m(x)) > \mathcal{M}^+(\tilde{C}_m(y)) \\
yPx & \iff \mathcal{M}^- (\tilde{C}_m(y)) > \mathcal{M}^+(\tilde{C}_m(x)) \\
xly & \text{ else.}
\end{align*}
\]

\((P, I)\) is an interval order ⇒ ranking on the alternatives of \(A\)

Particularity
\(I\) is not transitive.

For \(x, y, z \in A\) it is possible to have \(xly\) and \(ylz\), but \(xPz\)

Ranking and choosing by a complete fuzzy interval order

Credibility \(\Pi\) is a fuzzy interval order\(^4\) (Roubens and Vincke, 1988)

Roubens and Vincke define a preference relation by

\[
P(x, y) = \min(\Pi(x, y), 1 - \Pi(y, x)) \quad \forall (x, y) \in A^2.
\]

The associated digraph is called a fuzzy preference digraph

Score of non-domination of an alternative \(x\) of \(A\) (Orlovski, 1976)

\[
\text{ND}(x) = 1 - \max_{y \neq x} P(y, x).
\]

\(^4\) a reflexive, complete (\(xPy\) or \(yPx\) for each \(x \neq y\)) and Ferrers
(\(wPz \Rightarrow wPz\) or \(z\Pi x\)) valued relation

Core \(Y_0\) of \(A\): \(Y_0 \subset A\) such that its elements all have a score of non-domination of 1

The core is non-empty (Orlovski, 1976)

\(Y_0\) gives a solution to the choice problem

Ranking

Iterative procedure:

\(Y_1: \text{core of the subgraph } A \setminus Y_0\)

\(\ldots\)

Result: partition of \(A\), namely \((Y_i)_{i=0}^c\)

Total preorder \(Y_0 \succ Y_1 \succ \ldots \succ Y_c\) (the elements in each core are considered as indifferent)

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In practice

Partial evaluations of $x$ are trapezoidal fuzzy numbers
$\tilde{x}_i = (a_i, b_i, \alpha_i, \beta_i) \ (i \in N)$

Build a "simple" model: 2-additive Choquet integral

Aggregation:

$$\tilde{C}_m(x) = \sum_{i=1}^{n} m(\{i\}) \cdot \tilde{x}_i + \sum_{i,j=1}^{n} m(\{i,j\}) \cdot \tilde{\min}(\tilde{x}_i, \tilde{x}_j).$$

The min of two trapezoidal fuzzy numbers

Minimum of two trapezoidal fuzzy numbers can be summarised by 8 parameters $(a, b, \lambda^-, \alpha', \alpha'', \lambda^+, \beta', \beta'')$: 

![Figure: The minimum of two trapezoidal fuzzy numbers](image)

On the possibilistic means

$\tilde{m}$: minimum of two trapezoidal fuzzy numbers

We have:

$$\mathcal{M}^- (\tilde{m}) = a - \frac{\alpha' + \lambda^- \alpha''}{2} \quad \text{and} \quad \mathcal{M}^+ (\tilde{m}) = b + \frac{\beta' + \lambda^+ \beta''}{2}.$$ 

Furthermore, $\forall \tilde{x}, \tilde{y} \in \mathcal{F}$ and $\forall p \in \mathbb{R}$:

$$\mathcal{M}^\pm (p \cdot \tilde{x}) = p \mathcal{M}^\pm (\tilde{x})$$

$$\mathcal{M}^\pm (\tilde{x} + \tilde{y}) = \mathcal{M}^\pm (\tilde{x}) + \mathcal{M}^\pm (\tilde{y})$$

The possibilistic mean of the Choquet integral

For any alternative $x \in A$ we have:

$$\mathcal{M}^\pm [\tilde{C}_m(x)] = \sum_{i=1}^{n} m(\{i\}) \mathcal{M}^\pm (\tilde{x}_i) + \sum_{i,j=1}^{n} m(\{i,j\}) \mathcal{M}^\pm (\tilde{\min}(\tilde{x}_i, \tilde{x}_j)).$$

$\Rightarrow$ alternatives can easily be ordered in this particular case
Problems linked to the parameters of $v$ or $m$

Large number of parameters (even in a 2-additive Choquet integral)

Meaning of the parameters of $v$ or $m$??

Solution:
- determine the parameters of $v$ or $m$ by a learning procedure
- ask the decision maker only information from his domain of expertise

Prototype
A prototype is an alternative which is well-known for the decision maker, which is well defined and has crisp partial evaluations

We suppose that the decision maker can provide a set $P$ of prototypes

We suppose that he can determine a total order on the elements of $P$

The learning procedure: intuitively

Nothing guarantees that the total order on $P$ is compatible with a Choquet integral as a discriminant function

Therefore: determine a set function $m$ which will provide a satisfactory ranking on the prototypes

⇒ minimise the gap between the ranking of the decision maker and the one resulting from the aggregation

The learning procedure: in detail

The parameters of $m$ are determined by the following quadratic program:

$$\min \sum_{x \in P} [C_m(x) - y(x)]^2,$$

Unknowns:
- the parameters of the set function $m$;
- some global evaluations $y(x)$ for each $x \in P$.

Constraints:
- boundary and monotonicity conditions imposed on the fuzzy measure $v$
- the global evaluations $y(x)$ must verify the ranking imposed by the decision maker
### The learning procedure: the output

Parameters of $m$ and the global evaluations $y(x), \forall x \in A$

If the objective function equals 0:

- The ranking imposed by the decision maker corresponds to the ranking obtained by the aggregation

Else:

- In the ranking obtained by aggregation, certain pairs of alternatives are reversed compared to the ranking imposed by the decision maker

Adequation of both rankings ($R$ and $S$) (symetric difference)

$$\delta(R, S) = |\{(a, b) \in A^2 : (xRy \text{ and } \neg(xSy)) \text{ or } (\neg(xRy) \text{ and } xSy)\}|$$

### And then?

The output of the learning procedure is used to aggregate the fuzzy partial evaluations of the remaining alternatives

A ranking can be determined by either one of the three suggestions

### To do

- Test the method (implement it)
- Compare the three rankings