On the use of the Choquet integral with fuzzy numbers in Multiple Criteria Decision Aiding

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The Multiple Criteria Decision Aiding (MCDA) problematic

$A =$ set of alternatives

$N =$ set of points of view

Example:

$A =$ set of houses

$N = \{$price, surface area, age, $\ldots\}$

Each alternative is evaluated on each point of view

Objective

Build a ranking on the alternatives

A general case

The evaluations of the alternatives on the points of view can suffer from impreciseness

Example:

$A =$ set of establishment sites for a chemical industry

$N = \{$additional working places created, soil permeability, cost of establishement, $\ldots\}$

Hard to determine precise values for the alternatives on some points of view

For example: it is quite credible that "cost of establishement" lies approximatively in $[19 \cdot 10^6€, 21 \cdot 10^6€]$

Representation of imprecise information by fuzzy numbers

Fuzzy sets

Zadeh (1965) introduces the concept of fuzzy sets to represent impreciseness

Definition

A fuzzy subset $\widetilde{Y}$ of a classical set X is defined by its membership function

$$\mu_{\widetilde{Y}} : X \to [0,1].$$

For $x \in X$:

$\mu_{\widetilde{Y}}(x) = 0$: x does not belong to $\widetilde{Y}$
$\mu_{\widetilde{Y}}(x) = 1$: complete membership of x to $\widetilde{Y}$
$\mu_{\widetilde{Y}}(x) \in]0,1[$: intermediate memberships

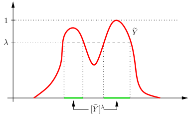
Notation: $\widetilde{Y}(x) := \mu_{\widetilde{Y}}(x)$

λ-level set of a fuzzy set $\widetilde{Y}$

Definition

$$[\widetilde{Y}]^\lambda = \begin{cases} \{x \in X : \widetilde{Y}(x) \geq \lambda\} & \text{if } \lambda > 0 \\ \text{cl}(\underline{\widetilde{Y}})^a & \text{if } \lambda = 0 \end{cases}$$

[a] closure of the support $\underline{\widetilde{Y}} = \{x \in X : \widetilde{Y}(x) > 0\}$



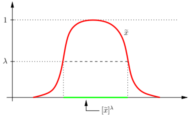
A λ-level set of a fuzzy set $\widetilde{Y}$ is also called a λ-cut of $\widetilde{Y}$

Fuzzy numbers

Definition

A fuzzy number $\widetilde{x}$ of $\mathbb{R}$ is a fuzzy subset of $\mathbb{R}$ that is normal[a], fuzzy convex[b] and has a continuous membership function of bounded support.

[a] $\exists x \in \widetilde{x}$ for which $\widetilde{x}(x) = 1$
[b] Any λ-cut is convex



Let $\mathcal{F}$ be the family of all fuzzy numbers

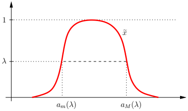
Fuzzy numbers

Definition

For $\widetilde{x} \in \mathcal{F}$:

$$a_m(\lambda) = \min[\widetilde{x}]^\lambda, \quad a_M(\lambda) = \max[\widetilde{x}]^\lambda$$

for each $\lambda \in [0, 1]$.



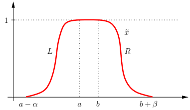
General way to write a fuzzy number

$$\widetilde{x}(x) = \begin{cases} L(\frac{a-x}{\alpha}) & \text{if } a - \alpha \leq x \leq a, \\ 1 & \text{if } a < x \leq b, \\ R(\frac{x-b}{\beta}) & \text{if } b < x \leq b + \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $[a, b]$ is the *peak* of $\widetilde{x}$,

$L, R : [0, 1] \rightarrow [0, 1]$ are upper semi-continuous and non-increasing functions, $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. L, R are called side functions.

The support $\underline{\widetilde{x}}$ is equal to $]a - \alpha, b + \beta[$.



Trapezoidal fuzzy numbers

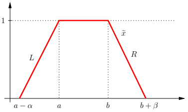
Very useful in Multiple Criteria Decision Aiding

$$\widetilde{x}(x) = \begin{cases} 1 - \frac{a-x}{\alpha} & \text{if } a - \alpha \leq x \leq a, \\ 1 & \text{if } a < x \leq b, \\ 1 - \frac{x-a}{\beta} & \text{if } b < x \leq b + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

α and β are called the left and right width,

notation $\widetilde{x} = (a, b, \alpha, \beta)$.

If $a = b$ the fuzzy number is called triangular and a is said to be the center of $\widetilde{x}$.



Possibility distributions and fuzzy numbers

Let $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $a \leq b$

Definition

The possibility that $\widetilde{x} \in \mathcal{F}$ takes its value from the interval $[a, b]$ is defined by

$$\text{Pos}(\widetilde{x} \in [a, b]) = \max_{x \in [a,b]} \widetilde{x}(x).$$

In particular for $\lambda \in [0, 1]$,

$$\text{Pos}(\widetilde{x} \leq a_m(\lambda)) = \max_{x \leq a_m(\lambda)} \widetilde{x}(x) = \lambda,$$

$$\text{Pos}(\widetilde{x} \geq a_M(\lambda)) = \max_{x \geq a_M(\lambda)} \widetilde{x}(x) = \lambda.$$

Zadeh's extension principle

Goal: To use mathematical operations on the fuzzy numbers, compatible with classical arithmetics

If $\widetilde{x}_1, \ldots, \widetilde{x}_n \in \mathcal{F}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then the sup-min extension principle of Zadeh is defined by

$$f(\widetilde{x}_1, \ldots, \widetilde{x}_n)(y) = \sup_{f(x_1,\ldots,x_n)=y} \min\{\widetilde{x}_1(x_1), \ldots, \widetilde{x}_n(x_n)\}, \forall y \in \mathbb{R}.$$

It has been shown that $f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is a fuzzy number.

Preliminary considerations

A = set of q alternatives

$N = \{1, \ldots, n\}$ set of labels of points of view

Partial evaluations of each alternative on each point of view made on an interval scale

$\Rightarrow$ admissible transformation function: positive linear transformation

Evaluations supposed to be commensurable

Performance scale of each point of view: $[0, 1]$

Preliminary considerations

Evaluations of the alternatives on the points of view are fuzzy numbers

Any alternative $x \in A$ can be identified with its corresponding fuzzy profile

$$x \equiv (\widetilde{x_1}, \ldots, \widetilde{x_n}) \in [0,1]^n$$

Objective

Rank the alternatives

1. Aggregate the partial evaluations of each alternative on the points of view in a global evaluation,
2. Order these global evaluations to obtain a ranking.

In general
Use of the weighted sum as an aggregator.

Disadvantage
Supposes preferential independence of the points of view

One possible solution
Use of the Choquet integral as an aggregator

Work hypothesis: preliminaries

Let $\widetilde{x_1}, \ldots, \widetilde{x_n} \in \mathcal{F}$

Definition

$\widetilde{c} \in \mathbb{R}^n$ is called the *joint possibility distribution* of $\widetilde{x_1}, \ldots, \widetilde{x_n}$ if it satisfies

$$\widetilde{c}(x_1, \ldots, x_n) \leq \min\{\widetilde{x_1}(x_1), \ldots, \widetilde{x_n}(x_n)\} \quad \forall x_i \in \mathbb{R}, i = 1, \ldots, n.$$

Equivalently: $[\widetilde{c}]^\gamma \subseteq [\widetilde{x_1}]^\gamma \times \ldots \times [\widetilde{x_n}]^\gamma \ \forall \gamma \in [0,1]$

Fuzzy numbers $\widetilde{x_i} \in \mathcal{F}$, $i = 1, \ldots, n$ are said to be non-interactive if $[\widetilde{c}]^\gamma = [\widetilde{x_1}]^\gamma \times \ldots \times [\widetilde{x_n}]^\gamma$ for all $\gamma \in [0,1]$

Zadeh implicitly supposes that the fuzzy numbers have to be non-interactive in the extension principle.

Work hypothesis: preliminaries

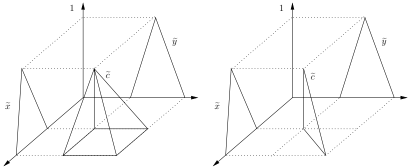


Figure: Non-interactive (left) and interactive (right) fuzzy numbers

Work hypothesis

Hypothesis

The partial evaluations of the alternatives are non-interactive (in the sense discussed before)

But:
Points of view can interact

- substitutivenes[1]
- complementarity[2]
- preferential dependence[3]

[1]The satisfaction of one point of view has almost the same effect as the satisfaction of both.

[2]The satisfaction of only one point of view is weak compared to the satisfaction of both points of view.

[3]Consider $x, y \in A$ for which the evaluations on $S \subseteq N$ are equal. The subset $N \setminus S$ is preferentially independent of S if the preference of x over y is not influenced by their common part on S.

The Choquet integral

$x \in A$, described by crisp partial evaluations $x_i, (i \in N)$

Definition

The Choquet integral of x is defined by:

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)}[v(A_{(i)}) - v(A_{(i+1)})]$$

where v represents a fuzzy measure on N, that is a monotone set function $v : 2^N \rightarrow [0,1]$ fulfilling $v(\emptyset) = 0$ and $v(N) = 1$.
The parentheses used for indexes represent a permutation on N such that

$$x_1 \leq \ldots \leq x_n,$$

and $A_{(i)}$ represents the subset $\{(i), \ldots, (n)\}$.

Advantages and drawback

Allows to represent interactions

Large number of parameters $(2^n - 2) \Rightarrow$ great flexibility

But: difficult to assign values to the parameters of v

Towards an equivalent representation

v fuzzy measure on N

The Möbius transform of v is a set function $m : 2^N \rightarrow \mathbb{R}$ defined by

$$m(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T) \ (S \subseteq N)$$

invertible transformation

v can be recovered from m

Choquet integral with m

m set function on N

Definition

$$\mathcal{C}_m(x) = \sum_{T \subseteq N} m(T) \bigwedge_{i \in T} x_i$$

Main difference: no reordering of the partial evaluations x_i

k-additivity

A fuzzy measure is k-additive if its Möbius transform m satisfies $m(S) = 0$ for S such that $|S| > k$ and there exists at least one subset S such that $|S| = k$ and $m(S) \neq \emptyset$.

Towards a fuzzy extension of the Choquet integral

Definition of the Choquet integral in terms of a set function m is a combination of functions which are continuous on $\mathbb{R} \times \mathbb{R}$: addition $(+)$, multiplication $(\cdot)$ and minimum $(\wedge)$

Extension principle of Zadeh: extend the three functions to their fuzzy versions

$\tilde{+}$

$\tilde{x}_1, \tilde{x}_2 \in \mathcal{F}$

$$\tilde{x}_1 \tilde{+} \tilde{x}_2(y) = \sup_{a+b=y} \min[\tilde{x}_1(a), \tilde{x}_2(b)]$$

Result: fuzzy number

Trapezoidal fuzzy numbers

$$\tilde{x} = (a_x, b_x, \alpha_x, \beta_x) \text{ and } \tilde{y} = (a_y, b_y, \alpha_y, \beta_y)$$
$$\tilde{x} \tilde{+} \tilde{y} = (a_x + a_y, b_x + b_y, \alpha_x + \alpha_y, \beta_x + \beta_y)$$

$\tilde{\cdot}$

$p \in \mathbb{R}$ (a coefficient of m), $\tilde{x}_1 \in \mathcal{F}$

$$[p \,\tilde{\cdot}\, \tilde{x}_1]^\lambda = p[\tilde{x}_1]^\lambda$$

for any $\lambda \in [0, 1]$

Result: fuzzy number

Trapezoidal fuzzy numbers

$$\tilde{x} = (a_x, b_x, \alpha_x, \beta_x)$$
$$p\tilde{x} = (pa_x, pb_x, p\alpha_x, p\beta_x).$$

$\widetilde{\wedge}$

$\widetilde{x_1}, \widetilde{x_2} \in \mathcal{F}$

$$\widetilde{\bigwedge}(\widetilde{x_1}, \widetilde{x_2})(y) = \sup_{\min(a,b)=y} \min[\widetilde{x_1}(a), \widetilde{x_2}(b)]$$

Result: fuzzy number

Trapezoidal fuzzy numbers

side functions become piecewise linear functions

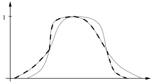


Figure: Minimum of two fuzzy numbers (bold dashes)

A fuzzy extension of the Choquet integral

Definition

$$\widetilde{\mathcal{C}_m}(x) = \widetilde{\sum_{T \subseteq N}} m(T) \,\tilde{\cdot}\, \widetilde{\bigwedge_{i \in T}} x_i.$$

where m is the set function which is obtained by a Möbius transform of the fuzzy measure v.

Result: fuzzy number

Trapezoidal fuzzy numbers

side functions become piecewise linear functions

Example

$N = \{1,2\}$, $v(\{1\}) = 0.1$, $v(\{2\}) = 0.2$, $v(\{1,2\}) = 1$

$m(\{1\}) = 0.1$, $m(\{2\}) = 0.2$, $m(\{1,2\}) = 0.7$

$\widetilde{x_1} = (1,1,0)$ and $\widetilde{x_2} = (0.5,0,1)$

$$\begin{aligned} \widetilde{\mathcal{C}_m}((\widetilde{x_1}, \widetilde{x_2})) &= m(\{1\}) \,\tilde{\cdot}\, \widetilde{x_1} \,\tilde{+}\, m(\{2\}) \,\tilde{\cdot}\, \widetilde{x_2} \,\tilde{+}\, m(\{1,2\}) \,\tilde{\cdot}\, \widetilde{\bigwedge}(\widetilde{x_1}, \widetilde{x_2}) \\ &= (0.2, 0.1, 0.2) \,\tilde{+}\, 0.7 \,\tilde{\cdot}\, \widetilde{\bigwedge}(\widetilde{x_1}, \widetilde{x_2}) \\ &:= \widetilde{s} \,\tilde{+}\, \widetilde{s'} \end{aligned}$$

where $\widetilde{s} = (0.2, 0.1, 0.2)$ and $\widetilde{s'} = 0.7 \,\tilde{\cdot}\, \widetilde{\bigwedge}(\widetilde{x_1}, \widetilde{x_2})$.

Example

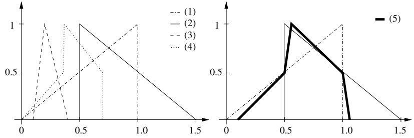


Figure: Example: towards the aggregation

(1) and (2) stand respectively for $\widetilde{x_1}$ and $\widetilde{x_2}$

(3) and (4) represent $\widetilde{s}$ and $\widetilde{s'}$

(5) shows the aggregated value $\widetilde{\mathcal{C}_m}((\widetilde{x_1}, \widetilde{x_2}))$

Possibilistic mean

Consider the membership function of a fuzzy number $\widetilde{x}$ as a possibility distribution

Upper possibilistic mean of $\widetilde{x}$

$$\mathcal{M}^+(\widetilde{x}) := \int_0^1 a_M(\lambda) d\lambda = \int_0^1 a_M(\lambda) d\text{Pos}(\widetilde{x} \geq a_M(\lambda))$$

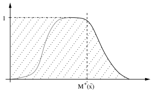


Figure: Representations of $\mathcal{M}^+(\widetilde{x})$: a number and a surface

Possibilistic mean

Lower possibilistic mean of $\widetilde{x}$

$$\mathcal{M}^-(\widetilde{x}) := \int_0^1 a_m(\lambda) d\lambda = \int_0^1 a_m(\lambda) d\text{Pos}(\widetilde{x} \leq a_m(\lambda)).$$

Possibilistic mean of $\widetilde{x}$

$$\mathcal{M}(\widetilde{x}) := \frac{1}{2}[\mathcal{M}^+(\widetilde{x}) + \mathcal{M}^-(\widetilde{x})].$$

Ranking by a complete preorder

Ranking by comparing the possibilistic means of the global evaluations of the alternatives of A

$x, y \in A$. Complete preorder $\succeq$ (*is not worse than*):

$$x \succeq y \iff \mathcal{M}(\widetilde{\mathcal{C}_m}(x)) \geq \mathcal{M}(\widetilde{\mathcal{C}_m}(y))$$

Geometric interpretation

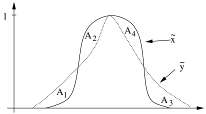


Figure: Comparing $\widetilde{x}$ to $\widetilde{y}$

It is possible to show that

$$x \succeq y \iff A_4 + A_1 \geq A_3 + A_2.$$

Corresponds to the area compensation method of Fortemps and Roubens (1996).

Ranking by a complete interval order

Ranking by comparing the intervals $[\mathcal{M}^-(\widetilde{\mathcal{C}_m}(x)), \mathcal{M}^+(\widetilde{\mathcal{C}_m}(x))]$

I symmetrical relation, P antisymmetrical relation on $A \times A$ such that

$$\begin{cases} xPy & \iff \mathcal{M}^-(\widetilde{\mathcal{C}_m}(x)) > \mathcal{M}^+(\widetilde{\mathcal{C}_m}(y)) \\ yPx & \iff \mathcal{M}^-(\widetilde{\mathcal{C}_m}(y)) > \mathcal{M}^+(\widetilde{\mathcal{C}_m}(x)) \\ xIy & \text{else.} \end{cases}$$

(P, I) is an interval order $\Rightarrow$ ranking on the alternatives of A

Particularity
I is not transitive.
For $x, y, z \in A$ it is possible to have xIy and yIz, but xPz

Ranking and choosing by a complete fuzzy interval order

Ranking based on a degree of credibility of the preference of one alternative on another one

Possibility Π that an alternative x is not worse than y (Fodor 1994) (let's write $x \succeq y$):

$$\Pi(x \succeq y) = \sup_{a \geq b}[\min(\widetilde{x}(a), \widetilde{y}(b))]$$

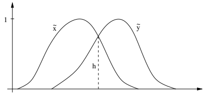


Figure: $\Pi(x \succeq y) = h$ and $\Pi(y \succeq x) = 1$

Ranking and choosing by a complete fuzzy interval order

Credibility Π is a fuzzy interval order[4] (Roubens and Vincke, 1988)

Roubens and Vincke define a preference relation by

$$P(x, y) = \min(\Pi(x, y), 1 - \Pi(y, x)) \quad \forall (x, y) \in A^2.$$

The associated digraph is called a fuzzy preference digraph

Score of non-domination of an alternative x of A (Orlovski, 1976)

$$ND(x) = 1 - \max_{y \neq x} P(y, x).$$

[4] a reflexive, complete ($x\Pi y$ or $y\Pi x$ for each $x \neq y$) and Ferrers ($w\Pi x, y\pi z \Rightarrow w\Pi z$ or $z\Pi x$) valued relation

Ranking and choosing by a complete fuzzy interval order

Core Y_0 of A: $Y_0 \subset A$ such that its elements all have a score of non-domination of 1

The core is non-empty (Orlovski, 1976)

Y_0 gives a solution to the *choice* problem

Ranking
Iterative procedure:
Y_1: core of the subgraph $A \setminus Y_0$
...
Result: partition of A, namely $(Y_i)_{i=0}^c$

Total preorder $Y_0 \succ Y_1 \succ \ldots \succ Y_c$ (the elements in each core are considered as indifferent)

In practice

Partial evaluations of x are trapezoidal fuzzy numbers $\widetilde{x_i} = (a_i, b_i, \alpha_i, \beta_i)$ $(i \in N)$

Build a "simple" model: 2-additive Choquet integral

Aggregation:

$$\widetilde{\mathcal{C}_m}(x) = \widetilde{\sum_{i=1}^{n}} m(\{i\}) \tilde{\cdot}\, \widetilde{x_i} \;\tilde{+}\; \widetilde{\sum_{\substack{i,j=1\\ i\neq j}}^{n}} m(\{i,j\}) \tilde{\cdot}\, \widetilde{\min}(\widetilde{x_i}, \widetilde{x_j}).$$

The min of two trapezoidal fuzzy numbers

Minimum of two trapezoidal fuzzy numbers can be summarised by 8 parameters $(a, b, \lambda^-, \alpha', \alpha'', \lambda^+, \beta', \beta'')$:

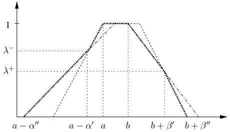


Figure: The minimum of two trapezoidal fuzzy numbers

On the possibilistic means

$\widetilde{m}$: minimum of two trapezoidal fuzzy numbers

We have:

$$\mathcal{M}^-(\widetilde{m}) = a - \frac{\alpha' + \lambda^-\alpha''}{2} \quad \text{and}$$

$$\mathcal{M}^+(\widetilde{m}) = b + \frac{\beta' + \lambda^+\beta''}{2}.$$

Furthermore, $\forall \widetilde{x}, \widetilde{y} \in \mathcal{F}$ and $\forall p \in \mathbb{R}$:

$$\mathcal{M}^\pm(p \,\tilde{\cdot}\, \widetilde{x}) = p\mathcal{M}^\pm(\widetilde{x})$$

$$\mathcal{M}^\pm(\widetilde{x} \,\tilde{+}\, \widetilde{y}) = \mathcal{M}^\pm(\widetilde{x}) + \mathcal{M}^\pm(\widetilde{y})$$

The possibilistic mean of the Choquet integral

For any alternative $x \in A$ we have:

$$\mathcal{M}^\pm[\widetilde{\mathcal{C}_m}(x)] = \sum_{i=1}^{n} m(\{i\})\mathcal{M}^\pm(\widetilde{x_i}) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} m(\{i,j\})\mathcal{M}^\pm(\min(\widetilde{x_i}, \widetilde{x_j})).$$

⇒ alternatives can easily be ordered in this particular case

Problems linked to the parameters of v or m

Large number of parameters (even in a 2-additive Choquet integral)

Meaning of the parameters of v or m??

Solution:

- determine the parameters of v or m by a learning procedure
- ask the decision maker only information from his domain of expertise

Hypothesis

Prototype

A prototype is an alternative which is well-known for the decision maker, which is well defined and has crisp partial evaluations

We suppose that the decision maker can provide a set P of prototypes

We suppose that he can determine a total order on the elements of P

The learning procedure: intuitively

Nothing guarantees that the total order on P is compatible with a Choquet integral as a discriminant function

Therefore: determine a set function m which will provide a satisfactory ranking on the prototypes

$\Rightarrow$ minimise the gap between the ranking of the decision maker and the one resulting from the aggregation

The learning procedure: in detail

The parameters of m are determined by the following quadratic program:

$$\min \sum_{x \in P} [\mathcal{C}_m(x) - y(x)]^2,$$

Unknowns:

- the parameters of the set function m;
- some global evaluations $y(x)$ for each $x \in P$.

Constraints:

- boundary and monotonicity conditions imposed on the fuzzy measure v
- the global evaluations $y(x)$ must verify the ranking imposed by the decision maker

The learning procedure: the output

Parameters of m and the global evaluations $y(x), \forall x \in A$

If the objective function equals 0:

the ranking imposed by the decision maker corresponds to the ranking obtained by the aggregation

Else:

in the ranking obtained by aggregation, certain pairs of alternatives are reversed compared to the ranking imposed by the decision maker

Adequation of both rankings (R and S) (symetric difference)

$$\delta(R,S) = |\{(a,b) \in A^2 : [xRy \text{ and } \neg(xSy)] \text{ or } [\neg(xRy) \text{ and } (xSy)]\}|,$$

And then?

The output of the learning procedure is used to aggregate the fuzzy partial evaluations of the remaining alternatives

A ranking can be determined by either one of the three suggestions

To do

Test the method (implement it)

Compare the three rankings