Exploitation of a bipolar-valued outranking relation for the choice of k best alternatives

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Abstract. This article presents the problem of the selection of k *best* alternatives in the context of multiple criteria decision aid. We situate ourselves in the context of pairwise comparisons of alternatives and the underlying bipolar-valued outranking digraph. We present three formulations for the best k-choice problem and detail how to solve two of them directly on the outranking digraph.

Mots-Clefs. best k-choice decision problem, multiple criteria decision aid, bipolar-valued outranking relation.

Introduction

In this article we discuss the problem of choosing k best decision objects (or alternatives) in the context of multiple criteria decision aid (*best k-choice problem*). We situate ourselves in the particular framework where the alternatives are compared pairwisely. Such a comparison produces a so-called bipolar-valued outranking relation on the set of alternatives which expresses the degree of confidence in the truth of a global pairwise preference situation.

The particular problem of selecting *one best* alternative ($k = 1$) is thoroughly discussed in [4]. In that particular work, the authors suggest a set of five pragmatic principles which underlie a progressive search (called the RuBy method) for one best alternative in a bipolar-valued outranking digraph. On basis of these principles, they determine the characteristics of the set of alternatives (namely the *hyperkernel*) of the digraph which must be retained for further considerations in the progressive search for the best decision object.

The work that we present in this article is based on the methodological studies of [4]. Nevertheless we extend the algorithm of the RuBy method to the problem of choosing k potentially best decision alternatives. As we show, this problem can have different formulations depending on the operational objectives.

The paper is organised as follows. In Section 1, we introduce the basic concepts and notations which will be necessary for our future discourse. Then, in Section 2, we present three formulations of the best k-choice problem. The third section deals with the resolution of two of these problems. In the fourth and last section, we present a small example which shows the differences between the two last formulations of the best k-choice problem.

1 Preliminary considerations ...

In order to clearly present the context in which this article is situated, we first define a certain number of useful concepts. For details on these notions, we advise the interested reader to refer to [4].

... on the decision aiding process

We focus on so-called *progressive* decision aiding processes. They allow to progressively and interactively reach the desired objective by refining the problem at different steps which require the interaction of the decision maker (DM). Two reasons motivate the use of such a progressive decision aiding process.

Firstly, it allows to treat the underlying problem with a lot of care in the sense that at each step of the refinement process, there must be well motivated reasons when it comes to drawing conclusions. For example, in the case of the search for one best alternative, at each step of the process, decision objects are removed from the set of potential best alternatives if and only if there are enough well motivated arguments which are in favour of such withdrawals.

Secondly, it allows to deal with problems with limited economic resources. Indeed, it is easily imaginable that at a given step of the decision process, all the decision objects cannot be evaluated on each of the ideally required criteria. In the context of a progressive approach it is nevertheless still possible to draw strong conclusions with the available information and to refine the evaluations at a later stage.

The search for a single best choice as defined in [4] clearly integrates into this type of approaches as it allows to explicit intermediate subsets of *best* decision objects which need to be reevaluated at each step of the progressive analysis. We call a *best choice recommendation* (BCR) such an intermediate recommendation. For the classical best choice decision problem, the ultimate BCR is a single alternative.

... on the bipolar-valued outranking relation

Let $X = \{x, y, z, \ldots\}$ be the finite set of $n \geq 2$ potential decision objects evaluated on a finite, coherent family $F = \{1, \ldots, p\}$ of $p \geq 2$ criteria. In order to model a *global pairwise preference situation* between any two alternatives of X, we use a *bipolar-valued outranking relation* as defined in [4].

Classically, an outranking situation xSy (x outranks y) between two decision alternatives $x, y \in X$ is assumed to hold if there is a sufficient majority of criteria which supports an "*at least as good as*" preferential statement and there is no criterion which raises a veto against it [10]. The formal definition given in [4] allows to assign a valuation $\widetilde{S} : X \times X \to \mathcal{L}$ (called the bipolar-valued characterisation) to each element of the outranking relation $S \subseteq X \times X$ and which takes its values in a so-called rational credibility scale $\mathcal{L} = [-1, 1]$. The semantics linked to the values of $\widetilde{S}$ are listed hereafter:

- $\widetilde{S}(x, y) = +1$ signifies that the statement "xSy" is *certainly true*;
- $\widetilde{S}(x, y) > 0$ signifies that statement "xSy" is *more true than false*. A sufficient majority of criteria warrants the truth of the outranking;
- $\widetilde{S}(x, y) = 0$ signifies that statement "xSy" is *logically undetermined*, i.e. could be *either true or false*;
- $\widetilde{S}(x, y) < 0$ signifies that assertion "xSy" is *more false than true*. There is only a minority of the criteria which warrants the truth of the outranking. This is equivalent to saying that a sufficient majority of criteria warrants the truth of the negation of the outranking;
- $\widetilde{S}(x, y) = -1$ signifies that assertion "xSy" is *certainly false*.

$\widetilde{S}(x, y)$ represents the degree of confidence in the truth of the outranking statement, for each $(x, y) \in X \times X$. $\widetilde{S}$ is called the bipolar-valued characterisation of S, or for short the *bipolar-valued outranking relation*.

The truthfulness of the disjunction (resp. the conjunction) of two logical statements on an outranking situation corresponds to the maximum (resp. the minimum) of their credibilities. The truthfulness of the negation of a logical statement corresponds to the opposite of its credibility. This establishes the median value 0 of $\mathcal{L}$ clearly as a so-called negational fixpoint [2, 3].

It is possible to recover the *crisp* outranking relation S as the set of pairs $(x, y) \in X \times X$ such that $\widetilde{S}(x, y) > 0$. The set X associated to the bipolar-valued characterisation $\widetilde{S}$ of the outranking relation $S \in X \times X$ is called the *bipolar-valued outranking digraph*, denoted $\widetilde{G}(X, \widetilde{S})$. We write $G(X, S)$ the corresponding so-called *strict 0-cut crisp outranking digraph* associated to $\widetilde{G}(X, \widetilde{S})$.

... on choices, kernels and hyperkernels

This subsection lists a certain number of definitions which lead to the concept of hyperkernel of a digraph.

A *path of order* $m \leq n$ in $\widetilde{G}(X, \widetilde{S})$ is a sequence $(x_i)_{i=1}^m$ of alternatives of X such that $\widetilde{S}(x_i, x_{i+1}) \geq 0$, $\forall i \in \{1, \ldots, m-1\}$. A *circuit of order* $m \leq n$ is a path of order m such that $\widetilde{S}(x_m, x_1) \geq 0$.

Definition 1. *An* odd chordless circuit $(x_i)_{i=1}^m$ *is a circuit of odd order* m *such that* $\widetilde{S}(x_i, x_{i+1}) \geq 0$, $\forall i \in \{1, \ldots, m-1\}$, $\widetilde{S}(x_m, x_1) \geq 0$ *and* $\widetilde{S}(x_i, x_j) < 0$ *otherwise.*

Note here that an odd chordless circuit may contain arcs which are in an undetermined state (and which may evolve later to a determined outranking situation in the progressive decision aid).

A choice in a given bipolar-valued outranking digraph is a non-empty subset of decision objects. A k-choice is a choice which contains k decision objects.

Definition 2.

1. *A choice* Y *in* $\widetilde{G}(X, \widetilde{S})$ *is said to be* outranking *(resp.* outranked*) if and only if* $x \notin Y \Rightarrow \exists y \in Y : \widetilde{S}(y, x) > 0$ *(resp.* $\widetilde{S}(x, y) > 0$*);*
2. Y *is said to be* independent *(resp.* strictly independent*) if and only if for all* $x \neq y$ *in* Y *we have* $\widetilde{S}(x, y) \leqslant 0$ *(resp.* $\widetilde{S}(x, y) < 0$*);*

Let us continue by the definition of the outranking and outranked neighbourhoods in a digraph.

Definition 3.

1. *The* outranking neighbourhood $\Gamma^+(x)$ *of a node (or equivalently an alternative)* x *of* X *is the union of* x *and the set of alternatives which are outranked by* x*;*
2. *The* outranking neighbourhood $\Gamma^+(Y)$ *of a choice* Y *is the union of the outranking neighbourhoods of the alternatives of* Y*;*
3. *The* private outranking neighbourhood $\Gamma_Y^+(x)$ *of an alternative* x *in a choice* Y *is the set* $\Gamma^+(x) \setminus \Gamma^+(Y \setminus \{x\})$.

For a given alternative x of a choice Y, the set $\Gamma_Y^+(x)$ represents the *personal* contribution of x to the outranking quality of Y. If the private outranking neighbourhood of x in Y is empty, this means that, when x is dropped from this choice, Y still remains an outranking choice. From this observation one can derive the following definition (which will be useful in Section 3).

Definition 4. *A choice* Y *is said to be* irredundant *if all the alternatives of* Y *have non-empty private neighbourhoods.*

Definition 5 introduces the main concepts developed in [4] which allow to progressively search for one best decision object. As we will see later in this article, these concepts are also useful in the context of the best k-choice decision problems.

Definition 5.

1. *An outranking (resp. outranked) and independent choice is called an* outranking *(resp.* outranked*)* kernel*;*
2. *An outranking (resp. outranked) and strictly independent choice is called a* determined *outranking (resp. outranked)* kernel*;*
3. *A choice* Y *in* $\widetilde{G}(X, \widetilde{S})$ *is said to be* hyperindependent *(resp.* strictly hyperindependent*) if and only if it consists of odd chordless circuits of order* $p \geq 1$ *which are independent (resp. strictly independent) of each other;*
4. *An outranking (resp. outranked) and hyperindependent (resp. strictly hyperindependent) choice is called an outranking (resp. outranked) hyperkernel (resp. determined hyperkernel).*

Note that in point 4 of Definition 5 above, singletons are assimilated to odd chordless circuits of order 1.

Classically, in the best choice methods like Electre I or Electre IS [9, 11], the outranking kernel(s) of an outranking digraph are taken as BCRs in the progressive search for one best alternative. Nevertheless, as it is shown in [4], the kernel may be too restrictive and in certain situations and either no recommendation may be performed, or obvious BCRs may be left out. In order to overcome this problem, the authors of [4] have defined the concept of hyperkernel as a generalisation of the classical kernel. They show in particular that an outranking hyperkernel can always be found in any bipolar-valued outranking digraph.

To conclude this subsection, we need to introduce the concept of valued kernels. A choice Y in $\widetilde{G}(X, \widetilde{S})$ may be characterised with the help of bipolar-valued membership assertions $\widetilde{Y} : X \to \mathcal{L}$, denoting the credibility of the fact that $x \in Y$ or not, for all $x \in X$. $\widetilde{Y}$ is called a *bipolar-valued characterisation* of Y, or for short a *bipolar-valued choice* in $\widetilde{G}(X, \widetilde{S})$. Similar semantics as for $\widetilde{S}$ (see above) can be recovered for $\widetilde{Y}$.

The following proposition, which establishes a link between the classical graph-theoretic and the algebraic representations of kernels has been proved in [5].

Proposition 1. *The outranking (resp. outranked) kernels of $\widetilde{G}(X, \widetilde{S})$ are among the bipolar-valued choices $\widetilde{Y}$ satisfying the respective following bipolar-valued kernel equation systems:*

$$\max_{y \neq x}[\min(\widetilde{Y}(y), \widetilde{S}(y, x))] = -\widetilde{Y}(x), \quad \textit{for all } x \in X; \tag{1}$$

$$\max_{y \neq x}[\min(\widetilde{S}(x, y), \widetilde{Y}(y))] = -\widetilde{Y}(x), \quad \textit{for all } x \in X. \tag{2}$$

In [5] it is also shown that a particular subset (namely the maximal sharp (determined) choices) of the solutions of the outranking (resp. outranked) kernel equation systems characterise the outranking (resp. outranked) (determined) kernels of $\widetilde{G}$.

In the case where the bipolar-valued outranking digraph contains odd chordless circuits, it may happen that no solution can be found to the outranking (resp. outranked) kernel equation systems. Nevertheless, as it is shown in [4], after applying the kernel equations to a proper modification of the original outranking digraph, they always provide at least one outranking (resp. outranked) hyperkernel.

. . . on the RuBy bcr

We now can turn to the RuBy bcr in the context of the search for one best alternative (1-best choice problem). As already mentioned, in [4] the authors list a set of five pragmatic principles which underlie a progressive search for one best decision object and which *characterise* the so-called RuBy bcr.

Theorem 1. *A choice in an outranking digraph $\widetilde{G}(X, \widetilde{S})$ is a* RuBy bcr *if and only if it is a maximally determined strict outranking hyperkernel.*

The determinateness of a choice Y in $\widetilde{G}(X, \widetilde{S})$ is given by the average value of the absolute values of the membership assertions of its bipolar-valued characterisation $\widetilde{Y}$. The strictness of the outranking of a choice is guaranteed if its determinateness as an outranking choice is bigger than its determinateness as an outranked choice.

Let us now illustrate the concepts of this first section on the following example (taken from [4]).

Example 1 *Let $\widetilde{G}_1(X_1, \widetilde{S}_1)$ be a bipolar-valued outranking digraph, where $X_1 = \{a, b, c, d, e\}$ and $\widetilde{S}_1$ is given in table 1 and the associated* strict 0-cut *crisp digraph $G_1(X, S)$ is represented in figure 1. Note the dashed arc from b to e which represents an undetermined outranking situation. At this stage of the progressive analysis, it could not yet be determined if b outranks e or not.*

The choices $\{a, b, e\}$, $\{b, c, d\}$, as well as $\{a, b, c\}$ for instance, are all outranking choices. $\{a, b, e\}$, $\{b, c, d\}$, $\{b, e, d\}$, and $\{a, c\}$ are irredundant outranking choices. Choice $\{a, b, d, e\}$ is an outranking hyperkernel. The undetermined outranking relation between d and e implies that the choice is not strictly hyperindependent. Note here that this potential bcr *would have been left out if the search was restricted to outranking kernels.*

$\widetilde{G}_1$ admits an outranking kernel $\{a, c\}$ and a hyperkernel $\{\{a, b, d\}, e\}$ which is both outranking and outranked, but not with the same degree of determinateness as we may see in Table 2. As hyperkernel $\{\{a, b, d\}, e\}$ more outranking than outranked, we finally have two potential bcr*s: $\{\{a, b, d\}, e\}$ and $\{a, c\}$. The first one is significantly more determined (0.5 against 0.2) than the second one. The* RuBy "best choice recommendation" *therefore is $\{\{a, b, d\}, e\}$, where alternative*

$\widetilde{S}_1$	a	b	c	d	e
a	1.0	0.2	-1.0	-0.7	-0.8
b	-0.6	1.0	0.8	1.0	0.0
c	-1.0	-1.0	1.0	0.2	0.8
d	0.6	-0.6	-1.0	1.0	-0.4
e	-1.0	-0.8	-0.4	-0.6	1.0

Table 1. Example 1: the bipolar-valued outranking relation

e is in an undetermined situation. In a future step of the decision aid, it might be determined whether e should be removed or not from the BCR.

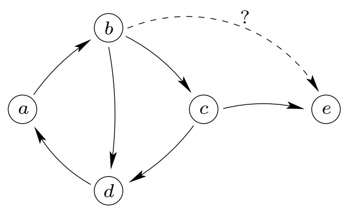

Fig. 1. Example 1: the associated strict 0-cut digraph and an undetermined arc

Y	$\{a,b,d\}$	a	b	c	d	e	D
$\{\{a,b,d\},e\}^+$	0.6	-0.6	-0.6	-0.6	-0.6	0.0	**0.5**
$\{a,c\}$	-0.2	0.2	-0.2	0.2	-0.2	-0.2	0.2
$\{\{a,b,d\},e\}^-$	0.0	0.0	0.0	-0.6	0.0	0.6	0.2

Table 2. Example 2: the characteristic vectors of the outranking ($^+$) and outranked ($^-$) hyperkernels.

This preliminary section on the main results of [4] was necessary as the future discourse requires the clear understanding of the concepts of RuBy BCR, bipolar-valued outranking relation and digraph. Furthermore, in Section 3 we show that the resolution of the best k-choice problem can be performed by searching for the RuBy BCR in an appropriately modified outranking digraph.

2 Different formulations of the best *k*-choice problem

A classical definition of the best 1-choice problem is given in [4]. From a pragmatic point of view, the goal of the best 1-choice problem is to select a unique best alternative. Nevertheless it is less obvious to give a single definition of the best k-choice problem. Indeed, different formulations to the selection of k best decision objects can be given:

$\mathcal{K}_1$ Search for the first k best alternatives (k first-ranked);
$\mathcal{K}_2$ Search for a set of k alternatives better than any other coalition of k alternatives (best k-team);
$\mathcal{K}_3$ Search for a set of k alternatives better than all the other alternatives (best k-committee).

Let us now detail these formulations one by one.

2.1 $\mathcal{K}_1$: considerations on the k first-ranked problem

This first formulation $\mathcal{K}_1$ corresponds probably to what people have commonly in mind when they hear about "*selecting the best k alternatives*" among a set of decision objects:

> Consider the k objects ranked in the first k positions of a total order or a total preorder (commonly called a ranking).

In case of multiple criteria decision aid, such a total order or total preorder can nevertheless only hardly be achieved in the framework of pairwise comparisons of alternatives. Indeed, the outranking relation which results from such pairwise comparisons is in general neither transitive nor total (some alternatives may be incomparable in terms of the outranking relation). Furthermore, in case the outranking relation is a partial order (or a partial preorder), it is difficult to conceive what the k first positions of the ranking could be.

These observations show that the outranking relation can difficultly be exploited directly to solve the problem of the k first-ranked alternatives. To overcome this problem, the outranking relation must first be exploited in order to build a total order or at least a preorder (see for example [1] or the Electre II, III and IV methods [11] or the Promethee I and II method [7, 6]). In that case, it is then possible to rank (possibly with ties) the alternatives from the *best* to the *worst* one (in terms of some measure derived from the outranking relation) and to select the k first ones. In case of ties in a preorder, selecting the k first ones might not be possible and it will be necessary to select $k' > k$ alternatives.

Another possibility to achieve total comparability of the alternatives in multiple criteria decision aid is via multi-attribute utility theory (MAUT)[1]. The aim of MAUT [8] is to model the preferences of the decision maker, represented by a binary relation $\succeq$, by means of an *overall utility function* $U : X \to \mathbb{R}$ such that,

$$x \succeq y \iff U(x) \geq U(y), \qquad \forall\, x, y \in X.$$

The preference relation $\succeq$ is assumed to be complete and transitive. This type of models generate a total order or a total preorder via the overall utilities. It is then possible to determine the k first-ranked alternatives in the same way as already mentioned earlier in the context of outranking methods.

From all the previous considerations, it is possible to derive a quite obvious, but nevertheless important and very general property:

Property 1. Let Y_i be the set of i first-ranked alternatives. $\forall k \leq n$, if Y_{k-1} and Y_k exist, we have

$$Y_{k-1} \subset Y_k.$$

This property is simply a translation of a quite natural intuition: the $k-1$ first-ranked alternatives also belong to the set of k first-ranked alternatives. This property is clearly verified in both cases described earlier. The possibly non-existence of Y_{k-1} or Y_k is simply due to the difficulty which arises in case of ties in total preorders.

It is obvious that in the case where $k = 1$, the k first-ranked problem amounts to selecting the first (and therefore best) alternative in the ranking.

Due to the necessity to exploit the outranking relation in order to obtain a total order or a total preorder, we will not explore this option further here. We will rathermore focus on the remaining two formulations $\mathcal{K}_2$ and $\mathcal{K}_3$ (which produce different results in general).

2.2 $\mathcal{K}_2$: introducing the best k-team problem

The second formulation of the best k-choice problem in a set X of alternatives can be summarised by the following intuitive procedure:

> Search for a set Y of cardinality k which *is better than* any other set of cardinality k.

[1] Here we briefly leave the framework of outranking methods. We nevertheless think that it quite natural to situate the k first-ranked problem in the context of MAUT.

The main difficulty lies in the formal definition of the "*is better than*"-relation for this particular case. Nevertheless, before dealing with this problem, let us first present practical situations in which the determination of the best k-team is applicable.

A first potential practical context is given by any situation where teams of k persons have to compete against each other and where each person has been evaluated individually on the family F of criteria. In our context, a pairwise comparison of all possible teams of k persons has then to be performed on basis of the individual outrankings.

A second kind of situations is given by facility location problems, where k locations have to be selected simultaneously. Again, any possible combination of k locations have to be pairwisely compared to determine which one is the most appropriate.

More generally, the best k-team problem is applicable in any situation where sets of alternatives have to be compared. Let us now turn to a more formal definition of the best k-team problem.

In the context of $\mathcal{K}_2$, a k-choice is called a k-team. Recall that we situate ourselves in a framework of pairwise comparisons of alternatives. Therefore it is quite natural to require that the following pragmatic conditions are verified by a k-team:

$\mathcal{T}_1$ **Inheritance**
A k-team inherits the outranking and outranked properties of its members;

$\mathcal{T}_2$ **Intra-team indiscernibility**
A k-team is considered as a unity from the *outside*;

$\mathcal{T}_3$ **Exclusive inter-team comparisons**
Two teams are exclusively compared on basis of information related to inter-team information.

The first property originates from the following observation. If an alternative $y \in X$ certainly outranks an alternative $y' \in X$ and if y and y' respectively belong to k-teams Y and Y', then this positive information for Y and negative information for Y' should be reflected in the way the two sets are compared.

The second property defines a very important characteristic of a k-team. The elements of a k-team should act together as a coalition and when compared to the other members of the team, a given alternative's weakness or strength should not be regarded. This clearly states the k-team as a coherent unity.

The third property is very important in the case where two teams which have a non-empty intersection are compared. In such a delicate situation, we require that the two sets of alternatives are only compared on basis of information which is not linked to their intersection. This will be clarified in the example which will be discussed hereafter.

Let us present a short example which allows to better understand the necessity of the three pragmatic principles $\mathcal{T}_1$ to $\mathcal{T}_3$.

Example 2 *Let us consider a set of 4 alternatives $X_2 = \{a, b, c, d\}$ and an outranking relation built from pairwise comparisons of these decision objects, $S_2 = \{(a, d), (b, c), (c, d)\}$. Recall that S_2 can be recovered from its bipolar-valued characterisation $\widetilde{S}_2$ (see Table 3 and Figure 2). Let us analyse how the three 2-teams $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ should be pairwisely compared.*

Sets $\{a, b\}$ and $\{b, c\}$ have alternative b in common, which is outranking alternative c. The arc between b and c is internal *to the set $\{b, c\}$. In accordance with principle $\mathcal{T}_3$, this information should not be taken into account when comparing $\{a, b\}$ and $\{b, c\}$. Therefore with the available outranking information, these two sets are incomparable.*

Sets $\{a, b\}$ and $\{a, c\}$ have alternative a in common. In that case, the arc between b and c is clearly inter-team *information and the set $\{a, b\}$ should outrank the set $\{a, c\}$.*

It is now clearer how the comparison of the sets of alternatives in the best k-team problem should be performed on basis of the outranking relation built on pairs of alternatives. A detailed analysis of this example is given in Section 4.

The three pragmatic properties above lead very naturally to the following literal definition of the outranking relation on the set of k-teams:

Definition 6. *Let $Y, Y' \subset X$ be two k-teams. Y outranks (resp. is outranked by) Y' if $\exists (y, y') \in (Y \times Y') \setminus (Y \cap Y')^2$ s.t. ySy' (resp. $y'Sy$).*

$\widetilde{S}_2$	a	b	c	d
a	1	< 0	< 0	> 0
b	< 0	1	> 0	< 0
c	< 0	< 0	1	> 0
d	< 0	< 0	< 0	1

Table 3. Example 2: Generic table representing $\widetilde{S}_2$

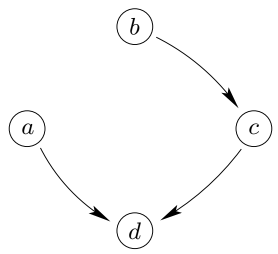

Fig. 2. Example 2: 0-cut outranking digraph

This definition clearly shows how the k-team should inherit the outranking and outranked characteristics of its elements, in accordance with principle $\mathcal{T}_1$. We will detail the construction of the outranking relation between pairs of sets of alternatives in Section 3. Note already at this point that due to principle $\mathcal{T}_2$, no condition is imposed on the incomparability (or independence) of the alternatives in a best k-team.

In Subsection 3.1 we will show how to solve the best k-team problem by basing our discourse on the previous pragmatic considerations, Definition 6 and the RuBy best choice method. To conclude this subsection, let us situate the best k-team problem in the context of progressive decision aid methods. In practice the ultimate objective is to determine a unique k-team which is considered as the best set of k alternatives. Nevertheless, as already mentioned in Section 1 it may be necessary to go through a few intermediate steps, where at each step, some k-teams are rejected for well motivated reasons. As we will show in Section 3, the problem of the best k-team can easily be solved by using the progressive RuBy method for the 1-best choice on a modified outranking digraph.

2.3 $\mathcal{K}_3$: introducing the best k-committee problem

In this subsection we start by giving a third intuitive definition of what the selection of k best alternatives could be:

> Search for a set Y of cardinality k which is in its entirety *better than* all the other alternatives.

Again, the problem here is to understand what the *better than*-relation signifies in this particular case. Similarly as for the $\mathcal{K}_2$ formulation, let us start by presenting a type of problem that the search for the best k-committee could address.

A potential practical context is given by any situation where in a set X of persons, a subset Y of k of them has to direct, pilot or command the remaining ones (for example a committee). In that case, each non-retained person of $X \setminus Y$ has to be considered as "*less preferred*" than Y in its collectiveness.

The main difference with the previous formulation $\mathcal{K}_2$ is that here, sets of alternatives have to be compared to single alternatives. In this context, a k-choice is called a k-committee. In the search for the best k-committee, we require that the following pragmatic principles are verified for any k-committee:

$\mathcal{C}_1$ Inheritance
A k-committee inherits the outranking and outranked properties of its members;

$\mathcal{C}_2$ **Intra-committee indiscernibility**
A k-committee should be considered as a unity from the *outside*;

$\mathcal{C}_3$ **Inter-committee comparisons**
k-committees are pairwisely compared via the alternatives they are outranking.

Principles $\mathcal{C}_1$ and $\mathcal{C}_2$ are similar to $\mathcal{T}_1$ and $\mathcal{T}_2$ for the best k-team problem of Subsection 2.2.

Principle $\mathcal{C}_3$ clearly shows the main difference between the best k-team and the best k-committee problem. Committees are compared via the single alternatives they outrank, whereas teams are compared to other teams.

Let us analyse on the reference example of Subsection 2.2 how committees behave in an outranking digraph.

Example 2 (continued) *Let us analyse how the three* 2*-committees* $\{a, b\}$*,* $\{a, c\}$ *and* $\{b, c\}$ *behave when compared to the remaining alternatives.*

Set $\{a, b\}$ *has to be compared to alternatives* c *and* d*.* c *is outranked by* b *and* d *is outranked by* a*. Therefore, the set* $\{a, b\}$ *should outrank both* b *and* d *(inheritance principle* $\mathcal{C}_1$*). Set* $\{b, c\}$ *has to be compared to alternatives* a *and* d*. A similar reasoning as before leads to saying that the set* $\{b, c\}$ *should outrank* d *but be incomparable to* a*. Finally, the set* $\{a, c\}$ *should be outranked by alternative* b *and outrank alternative* d*.*

Once again, a detailed analysis of this example in the case of the problem of searching for the best k-committee is presented in Section 4.

The three pragmatic principles lead to a definition of the outranking relation in the case of the k-committee problem.

Definition 7. *Let* $Y \subset X$ *be a* k*-committee and* $x \in X$*.* Y *outranks* x *if* $\exists y \in Y$ *s.t.* ySx*.*

Again, this definition shows that a k-committee inherits the outranking and outranked characteristics of its elements. In Section 3 we will detail the construction the outranking relation between sets of alternatives and single ones. Note again that no condition is given on the incomparability (or independence) of the alternatives in a k-committee. Similarly as for the best k-team problem, it is once again possible to situate the best k-committee problem in the framework of progressive outranking methods.

This section has presented three formulations for the best k-choice decision problem in the framework of outranking methods. In Subsection 2.1 we explained why we focus on the two problems $\mathcal{K}_2$ and $\mathcal{K}_3$ in this paper. In the following section we detail the resolution of the two best k-choice problems by means if the RuBy best choice method.

3 Solving $\mathcal{K}_2$ and $\mathcal{K}_3$

In this section we present how both formulations $\mathcal{K}_2$ and $\mathcal{K}_3$ of the best k-choice problem can be solved by using the progressive RuBy best choice decision method. In both cases the bipolar-valued outranking digraph, built on the set of alternatives, needs to be modified to obtain the desired BCR. The main motivation to use the RuBy method because it has strong pragmatic foundations which are developed in [4].

3.1 $\mathcal{K}_2$: best k-team

Recall that the goal of the best k-team problem is to select a set of k alternatives which is better than any other set of k alternatives. In view of the discussions of Subsection 2.2 one can easily understand that the best k-team problem can be solved in an outranking digraph $\widetilde{G}^t(X^t, \widetilde{S}^t)$ where the nodes represent all possible sets of k alternatives.

If the outranking relation $\widetilde{S}^t$ is appropriately defined (see Definition 6) , then the progressive search for the best k-team amounts to the progressive search for the RuBy BCR in that new digraph. Let us now detail the construction of $\widetilde{G}^t$.

The nodes X^t of $\widetilde{G}^t$ represent all possible subsets of k alternatives of X. The cardinality of X^t is therefore $\binom{n}{k}$ (which might be quite large for certain combinations of n and k). We label the nodes of X^t by capital letters in the sequel[2] as they represent subsets of X.

The outranking relation $\widetilde{S}^t$ is built as follows (based on Definition 6):

Let $Y, Y' \subset X$ be two k-teams. Y outranks (resp. is outranked by) Y' if $\exists (y, y') \in (Y \times Y') \setminus (Y \cap Y')^2$ s.t. ySy' (resp. $y'Sy$).

$$\begin{aligned} &\forall (V, W) \in X^t \times X^t : \\ &\quad \widetilde{S}^t(V, W) = \max\{\widetilde{S}(v, w) : (v, w) \in (V \times W) \setminus (V \cap W)^2\}. \end{aligned} \tag{3}$$

The crisp outranking relation S^t associated to $\widetilde{S}^t$ can be recovered as the set of pairs $(V, W) \in X^t \times X^t$ such that $\widetilde{S}^t(V, W) > 0$.

As mentionned in Section 1, the max operator models the truthfulness of the disjunction of logical statements. In fact, for a k-team Y to outrank another k-team Y', it is sufficient that one alternative of Y (positively) outranks another alternative of Y'. Furthermore, this aggregation of the outrankings allows to model complementarity among the different alternatives of a k-team and the alternatives of a k-team are considered as a unity. All in all, the construction of $\widetilde{S}^t$ as detailed in formula 3.1 clearly satisfies principles $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

It is now obvious that all the concepts introduced in Section 1 can be used in $\widetilde{G}^t(X^t, \widetilde{S}^t)$ and have a signification in $\widetilde{G}(X, \widetilde{S})$ in terms of subsets of alternatives. For example, a hyperindependent choice in $\widetilde{G}^t(X^t, \widetilde{S}^t)$ is a choice in $\widetilde{G}(X, \widetilde{S})$ which is composed of independent odd chordless circuits of subsets of X.

The objective of $\mathcal{K}_2$ – select a set of k alternatives which is better than any other set of k alternatives – can now be reinterpreted in $\widetilde{G}^t(X^t, \widetilde{S}^t)$. The goal of $\mathcal{K}_2$ in $\widetilde{G}^t(X^t, \widetilde{S}^t)$ is to select one unique node which is considered as the best one. This definition is very comparable to the search for one best alternative in an outranking digraph (see Section 2 or [4]).

Consequently, in the context of a progressive approach for the determination of the best k-team, the solution is to apply the RuBy method to the digraph $\widetilde{G}^t(X^t, \widetilde{S}^t)$. As already mentioned, it will exploit the bipolar-valued outranking relation in order to extract at least one maximally determined strict outranking hyperkernel (the BCR). The elements of this hyperkernel are subsets of k elements of X which are incomparable (or considered as equivalent in an odd chordless circuit).

In the case where the BCR is unique and only contains one element V of $\widetilde{G}^t(X^t, \widetilde{S}^t)$, then the problem is solved and V is a subset of k alternatives of X which can be considered as the best k-team. If the BCR contains more than one element of $\widetilde{G}^t(X^t, \widetilde{S}^t)$, then these k-teams should not be considered as the *best* ones, but merely as a collection of hardly comparable subsets of alternatives, among which the best k-team can be found (the not selected k-teams have been rejected for a well motivated reason). Similarly, in the case of multiple BCRs of equal determinateness, it is recommendable to continue the progressive approach with the union of the elements of the BCRs. Indeed, the only certain information is that some k-teams could be set aside for well motivated reasons. In the next step of the progressive approach the decision maker can restrict his analysis to these potential k-teams and refine their evaluations or evaluate their members on further criteria. At this step, the rejected subsets of k alternatives are rejected for well motivated reasons and can be left out without any regret. The ultimate step of this progressive approach will then be the determination of the best k-team.

One can easily see that the search for one best alternative as defined in [4] is a particular case of the best k-team problem for which $k = 1$.

3.2 $\mathcal{K}_3$: best k-committee

Recall that the goal of the search for the best k-committee is to determine a set Y of cardinality k which is in its entirety better than all the other alternatives. The problem will this time again be solved in a modified bipolar-valued outranking digraph $\widetilde{G}^c(X^c, \widetilde{S}^c)$, but this time its construction is less obvious than for $\mathcal{K}_2$. We will nevertheless show in this subsection that after a proper construction of $\widetilde{G}^c$, the problem of finding the best k-committee is again a particular case of the RuBy method in that new digraph.

[2] For the sake of simplicity, an element Z of X^t will represent a node of X^t as well as a subset of X.

As this time the comparison is between sets of nodes and single alternatives, the set X^c is defined as the union of X and a set of supplemental nodes which represent all possible subsets of k nodes of X. We use the same conventions as for $\mathcal{K}_2$ (see footnote 2) and therefore label the supplemental nodes (called k-nodes) in X^c by capital letters (and the nodes of X in X^c are still labelled with lower case letters).

The construction of $\widetilde{S}^c$ is somehow trickier. The original relation $\widetilde{S}$ is not included into $\widetilde{S}^c$. Then $\widetilde{S}^c$ is built as follows:

$$\begin{aligned}
\forall (V,W) \in X^c \times X^c : \quad & \widetilde{S}^c(V,W) = 0; & (1)\\
\forall (V,w) \in X^c \times X^c \text{ s.t. } w \in V : \quad & \widetilde{S}^c(V,w) = 1 \text{ and } \widetilde{S}^t(w,V) = 1; & (2)\\
\forall (V,w) \in X^c \times X^c \text{ s.t. } w \notin V : \quad & \widetilde{S}^c(V,w) = \max\{\widetilde{S}(v,w) : v \in V\}; & (3)\\
\forall (v,W) \in X^c \times X^c \text{ s.t. } v \notin W : \quad & \widetilde{S}^c(v,W) = \max\{\widetilde{S}(v,w) : w \in W\}. & (4)\\
\forall (v,w) \in X^c \times X^c : \quad & \widetilde{S}^c(v,w) = 0 & (5)
\end{aligned}$$

Let us explain this construction in further details. Formulae (1) and (5) puts any two k-committees and any two single alternatives in an undetermined situation. Two k-committees are compared via the alternatives they outrank. This motivates why they are pairwisely put in undetermined situations. As we will see later in the resolution algorithm, this is necessary for the first step which is a filtering stage. A similar reasoning justifies why pairs of alternatives are also put in an undetermined situation. In Formula (2) k-committees are quite naturally considered as equivalent to their members. Formulae (3) and (4) allow the comparison of k-committees to the remaining alternatives, pursuant to Definition 7.

Let us detail the determination of the best k-committee, at a given step of the progressive search. Recall that the best k-committee is a set of k alternatives which is in its entirety better than all the other ones. In $\widetilde{G}^c$ this amounts to searching for at least one k-node V which outranks all the alternatives $x \in X$.

The algorithm for the progressive search of the k-committee is a 2-step one:

Algorithm

Input: $\widetilde{G}^c(X^c, \widetilde{S}^c)$

1. *Search for the set $\mathcal{I}^c$ of irredundant outranking choices of $\widetilde{G}^c(X^c, \widetilde{S}^c)$ containing exclusively k-nodes;*
2. $\forall\ Y^c \in \mathcal{I}^c$*:*
   - *Remove any k-nodes from X^c which are not in $\mathcal{I}^c$ ($:= X^c_*$);*
   - *if $|Y^c| = 1$ then determine the* RuBy *best choice in $\widetilde{G}^c_*(X^c_*, \widetilde{S}^c)$;*
   - *else :*
     i. *modify $\widetilde{S}^c$ as follows into $\widetilde{S}^c_*$:*

$$\begin{aligned}
\widetilde{S}^c_*(V,W) = -1 \qquad & \forall (V,W) \in Y^c \times Y^c;\\
\widetilde{S}^c_*(x,y) = \widetilde{S}^c(x,y) \qquad & \textit{else.}
\end{aligned}$$

     ii. *Determine the* RuBy *best choice in $\widetilde{G}^c_*(X^c_*, \widetilde{S}^c_*)$ (containing exclusive k-nodes);*
3. *Select the most determined bipolar-valued* RuBy *best choice(s) among all those determined at step 2;*

Output: *a single (resp. a set of)* RuBy BCR*(s).*

In the present context, the output RuBy BCRs each contains independent potential best k-committees.

Let us analyse this algorithm in further details. Due to the particular way we construct $\widetilde{G}^c$ (and in particular $\widetilde{S}^c$), the output of the first step is one or more irredundant outranking choices containing exclusively k-nodes (the potential candidates for the best k-committee). This shows that the first stage of the algorithm is used for filtering purposes. In the second step, the RuBy best choice algorithm is applied to a modified graph for each irredundant outranking choice determined in the first step. Each graph $\widetilde{G}^c_*$ is composed of the original alternatives and an irredundant outranking choice. The modification of the outranking relation consists in removing the undetermined arcs that link the potential k-committees in the outranking choice. This allows the RuBy algorithm to determine the desired strict BCR.

Similarly as earlier, if the output is not a single k-node, then the progressive search must be reapplied to the set of potential best k-committees (and the set of alternatives which compose the k-committees).

4 Example

In this section we develop a detailed description of the example presented in Section 2. In order to simplify the notations, we will label the alternatives of X^t for example by concatenations of the labels of the alternatives of X. For example, the node of X^t representing the subset $\{a, b, c\}$ of X will be labelled abc.

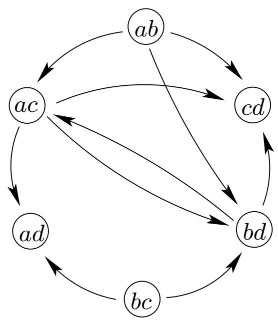

Fig. 3. 0-cut outranking digraph for the best 2-team problem

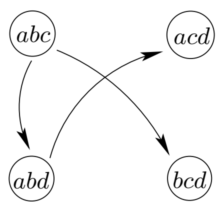

Fig. 4. 0-cut outranking digraph for the best 3-team problem

Two possible best k-team searches can be performed on this example (namely for $k = 2$ and $k = 3$). Both situations are represented on Figures 3 and 4.

The RuBy BCR for the 2-team problem is given by the set $\{\{a, b\}, \{b, c\}\}$. These two potential candidates as a best 2-team are incomparable and are therefore selected for a further analysis. This signifies that in the next step of the progressive approach, the decision maker can focus on these two subsets of alternatives in order to determine which one is the best one. The other subsets of 2 alternatives can already be rejected without any regret at this stage of the progressive decision aiding process.

The RuBy BCR for the 3-team problem is given by the set $\{\{a, b, c\}, \{a, c, d\}\}$. Again, these two potential candidates for a best 3-team are incomparable and the best candidate might be found in a further step of the decision aiding process.

In case of the search for the best k-committee, again two searches can be performed (namely for $k = 2$ and $k = 3$). Both situations are represented on Figures 5 and 6. The dotted (resp. dashed) arcs represent the 0-cut relations of type (1) (resp. (2)) from the definition of $\widetilde{S}^c$. As one can clearly see, they are merely technical arcs to allow the use of the RuBy BCR algorithm.

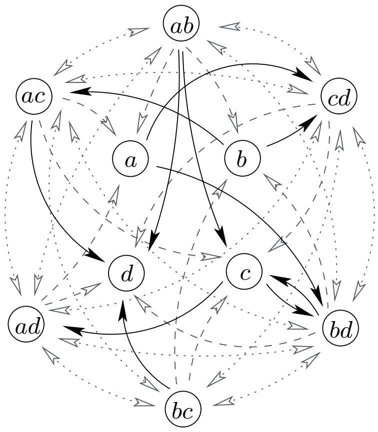

Fig. 5. 0-cut outranking digraph for the best 2-committee problem

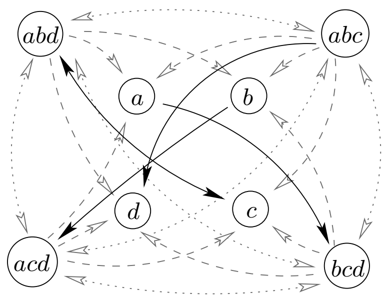

Fig. 6. 0-cut outranking digraph for the best 3-committee problem

The RuBy BCR for the 2-committee problem is given by the choice $\{a, b\}$. Indeed, the set $\{a, b\}$ clearly respects the definition of the 2-committee, namely that both alternatives are outranking in their entirety c and d.

For the best 3-committee problem, two potential k-sets could be considered: either the choice $\{a, b, c\}$ or the choice $\{a, b, d\}$. The final selection of either one (or both) of these choices as BCR(s) will depend on their determinateness and / or their strictness. Both concepts directly depend on the precise values of the bipolar-valued characterisations of the two potential choices.

Concluding remarks

This article presents our work on the problem of the selection of k *best* alternatives in the context of multiple criteria decision aid in presence of a bipolar-valued outranking relation. We show that there are (at least) three formulations to the best k-choice decision problem and give solutions for two of them.

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