4OR manuscript No.
(will be inserted by the editor)

RUBIS : a bipolar-valued outranking method for the choice problem

Raymond Bisdorff[1], **Patrick Meyer**[1⋆], **Marc Roubens**[2]

[1] Applied Mathematics Unit, University of Luxembourg, 162a, avenue de la Faïencerie, L-1511 Luxembourg, G.D. Luxembourg
[2] Faculté Polytechnique de Mons, 9, rue de Houdain, B-7000 Mons, Belgium

Abstract The main concern of this article is to present the RUBIS method for tackling the choice problem in the context of multiple criteria decision aiding. Its genuine purpose is to help a decision maker to determine a single best decision alternative. Methodologically we focus on pairwise comparisons of these alternatives which lead to the concept of bipolar-valued outranking digraph. The work is centred around a set of five pragmatic principles which are required in the context of a progressive decision aiding methodology. Their thorough study and implementation in the outranking digraph lead us to define a choice recommendation as an extension of the classical digraph kernel concept.

Key words: choice problematique, multiple criteria outranking method, progressive decision aiding methodology, digraph kernel.
MSC classification: 05C20, 90B50.

1 Introduction

We present a new method for constructing *choice recommendations* in the context of *multiple criteria decision aiding* where the objective is to determine a *single best alternative* from a set of potential decision objects. This work is situated in the context of *progressive* decision aiding methods (see Section 3.1) consisting normally in several stages providing the decision maker (DM) with more and more precise choice recommendations. Each of

⋆ Corresponding author: Patrick Meyer, Applied Mathematics Unit, University of Luxembourg, e-mail: patrick.meyer@uni.lu.

these steps aims at determining a subset of alternatives among which the best one is situated. As long as such a provisional recommendation consists of several candidates, the decision aiding process may be restarted on this restricted set with new and more detailed information in order to further assist the DM in his decision problem. Note that it may be up to him to determine the ultimate choice from the eventual recommendation of the decision aiding.

Apart from the European multiple criteria decision aiding community (Roy, 1985; Roy and Vanderpooten, 1996), the progressive resolution of the choice problem has attracted quite little attention by the Operational Research (OR) field. Seminal work on it goes back to the first article of Roy on the Electre I methods (Roy, 1968). After Kitainik (1993), interest in solving the choice problem differently from the classical optimisation paradigm has reappeared. An early work of Bisdorff and Roubens (1998) on valued kernels has resulted in new attempts to tackle the progressive choice problem directly on the valued outranking digraph. After first positive results (Bisdorff, 2000), methodological difficulties appeared when facing highly non-transitive and partial outranking relations. In this paper, we therefore present a new proposal for computing provisional choice recommendations from a valued outranking digraph. Our approach is based on new pragmatic and logical foundations of the progressive choice problematique[1] in the tradition of the pioneering work of Roy and Bouyssou (1993).

The paper is organised as follows. In Section 2, we introduce the basic concepts and notation which are necessary for our future discourse. In Section 3, we revisit the very foundations of the choice problematique, briefly present the classical Electre methodology, and list new pragmatic principles which are required for computing choice recommendations in a progressive decision aiding process. The fourth section deals with the translation of these principles into properties in the bipolar-valued outranking digraph which lead to the concept of hyperkernel, an extension of the classical kernel of a digraph. In the fifth and last section, we show how to determine these hyperkernels and detail the Rubis method[2] for computing a choice recommendation.

2 Fundamental concepts

We start by establishing the backbone of the Rubis method, namely the bipolar-valued credibility scale, modelling the credibility of the validation of preferential statements.

[1] A broad typology or category of problems for which multiple criteria decision analysis may be useful (Belton and Stewart, 2002, page 15).

[2] We propose the name Rubis as a generic identifier for bipolar-valued concordance-based decision aiding methods.

2.1 Bipolar-valued credibility calculus

Let ξ be a propositional statement like "*alternative x is a choice recommendation*" or "*alternative x is at least as good as alternative y*". In a decision process, a DM may either accept or reject these statements following his belief in their validation (Bisdorff, 2000). This degree of credibility (or *credibility* for short) may be represented via a credibility scale $\mathcal{L} = [-1, 1]$ supporting the following semantics. Let ξ and ψ be two propositional statements to which are associated credibilities r and $s \in \mathcal{L}$:

1. If $r = +1$ (resp. $r = -1$) then it is assumed that ξ is *clearly validated* (resp. *clearly non-validated*). If $0 < r < +1$ (resp. $-1 < r < 0$) then it is assumed that ξ is *more validated than non-validated* (resp. *more non-validated than validated*). If $r = 0$ then ξ could either be validated or non-validated, a situation we call *undetermined*.
2. If $r > s$ then it is assumed that the *validation* of ξ is more credible than that of ψ (or that the non-validation of ψ is more credible than that of ξ).
3. The credibility of the *disjunction* $\xi \vee \psi$ (resp. the *conjunction* $\xi \wedge \psi$) of these statements equals the credibility of the statement that is the most (resp. the less) credible of both, i.e. $\max(r, s)$ (resp. $\min(r, s)$).
4. $-r \in \mathcal{L}$ denotes the credibility of the non-validation of ξ, i.e., the credibility of the validation of the *logical negation* of ξ (written $\neg\xi$).

Definition 2.1. *The credibility associated with the validation of a propositional statement ξ, defined on a credibility domain $\mathcal{L}$ and verifying properties (1) to (4) is called a* bipolar-valued characterisation of ξ.

It follows from property (4) that the graduation of credibility degrees concerns both the *affirmation* and the *negation* of a propositional statement (Windelband, 1884). Starting from $+1$ (certainly validated) and -1 (certainly non-validated), one can approach a central position 0 by a gradual weakening of the absolute values of the credibility degrees. This particular point 0 in $\mathcal{L}$ represents an *undetermined* situation concerning the validation or non-validation of a given propositional statement (Bisdorff, 2000, 2002).

Definition 2.2. *The* degree of determination *of the validation (for short* determinateness*) $D(\xi)$ of a propositional statement ξ is given by the absolute value of its bipolar-valued characterisation: $D(\xi) = |r|$.*

For both a clearly validated and a clearly non-validated statement, the determinateness equals 1. On the opposite, for an undetermined statement, this determinateness equals 0.

This establishes the central degree 0 as an important *neutral* value in the bipolar-valued credibility calculus. Propositions characterised with this degree 0 may be either seen as *suspended* or as *missing* statements (Bisdorff, 2002). The credibility degree 0 represents a temporary delay in characterising the validation or non-validation of a propositional statement. In the framework of progressive decision aiding, this feature allows us to easily cope with currently undetermined preferential situations that may eventually become determined to a certain degree, either as validated or non-validated, in a later stage of the decision aiding process.

The following subsection introduces the important concept of bipolar-valued outranking digraph which is the preferential support for the RUBIS choice decision aiding methodology.

2.2 The bipolar-valued outranking digraph

Let $X = \{x, y, z, \ldots\}$ be a finite set of decision objects (or alternatives) evaluated on a finite, coherent family $F = \{1, \ldots, p\}$ of p criteria. To each criterion j of F is associated its *relative significance weight* represented by a rational number w_j from the open interval $]0, 1[$ such that $\sum_{j=1}^{p} w_j = 1$. Besides, to each criterion j of F is linked a preference scale in $[0, 1]$ which allows to compare the performances of the decision objects on the corresponding preference dimension.

Let $g_j(x)$ and $g_j(y)$ be the performances of two alternatives x and y of X on criterion j. Let $\Delta_j(x, y)$ be the difference of the performances $g_j(x) - g_j(y)$. To each preference scale for each j of F is associated a variable indifference threshold $q_j(g_j(x)) \in [0, 1[$, a preference threshold $p_j(g_j(x)) \in [q_j(g_j(x)), 1[$, a weak veto threshold $wv_j(g_j(x)) \in [p_j(g_j(x)), 1] \cup \{2\}$ and a strong veto threshold $v_j(g_j(x)) \in [wv_j(g_j(x)), 1] \cup \{2\}$, where the complete absence of veto is modelled via the value 2. All these threshold functions are supposed to verify the standard non-decreasing monotonicity condition (Roy and Bouyssou, 1993, page 56).

Let S be a binary relation on X. Classically, an outranking situation xSy between two decision alternatives x and y of X is assumed to hold if there is a sufficient majority of criteria which supports an "*at least as good as*" preferential statement and there is no criterion which raises a veto against it (Roy, 1985). The validation of such an outranking situation may quite naturally be expressed in the bipolar credibility calculus defined in Subsection 2.1. Our formulation is based on the classical ELECTRE definition of the outranking index. Nevertheless the reader should notice some slight but important differences, due to the semantics of the underlying bipolar valuation.

Indeed, in order to characterise a local "*at least as good as*" situation between two alternatives x and y of X for each criterion j of F, we use the following function $C_j : X \times X \rightarrow \{-1, 0, 1\}$ such that:

$$C_j(x,y) = \begin{cases} 1 & \text{if} \quad \Delta_j(x,y) > -q_j(g_j(x))\,; \\ -1 & \text{if} \quad \Delta_j(x,y) \leqslant -p_j(g_j(x))\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Credibility 0 is assigned to $C_j(x,y)$ in case it cannot be determined whether alternative x is at least as good as alternative y or not (see Subsection 2.1). Note here the deviation from the classical ELECTRE concordance index, where a linear transition between certainly validated and certainly non-validated situations is proposed (Roy and Bouyssou, 1993).

Similarly, the local veto situation for each criterion j of F is characterised via a veto-function $V_j : X \times X \rightarrow \{-1, 0, 1\}$ where:

$$V_j(x,y) = \begin{cases} 1 & \text{if} \quad \Delta_j(x,y) \leqslant -v_j(g_j(x))\,; \\ -1 & \text{if} \quad \Delta_j(x,y) > -wv_j(g_j(x))\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Again, according to the semantics of the bipolar-valued characterisation, the veto function V_j renders an undetermined response when the difference of performances is between the weak and the strong veto thresholds wv_j and v_j.

The global outranking index $\widetilde{S}$, defined for all pairs of alternatives $(x,y) \in X \times X$, conjunctively combines a global concordance index, aggregating all local "*at least as good as*" statements, and the absence of veto on each of the criteria. For any two alternatives x and y of X we have:

$$\widetilde{S}(x,y) = \min\{\widetilde{C}(x,y), -V_1(x,y), \ldots, -V_p(x,y)\}, \qquad (2.1)$$

where the global concordance index $\widetilde{C}(x,y)$ is defined as follows:

$$\widetilde{C}(x,y) = \sum_{j \in F} w_j \cdot C_j(x,y). \qquad (2.2)$$

The min operator in Formula 2.1 translates the conjunction between the global concordance index $\widetilde{C}(x,y)$ and the *negated* criterion-based veto indexes $-V_j(x,y)$ $(\forall j \in F)$. If all the V_j are equal to -1 (no veto situation is observed), the resulting outranking index $\widetilde{S}$ equals the global concordance index $\widetilde{C}$. Following Formulae (2.1) and (2.2), $\widetilde{S}$ is a function from $X \times X$ to $\mathcal{L}$ representing the credibility of the validation or non-validation of an outranking situation observed between each pair of alternatives. $\widetilde{S}$ is called the bipolar-valued characterisation of the outranking relation S, or for short, the *bipolar-valued outranking relation*.

The maximum value +1 of the valuation is reached in the case of unanimous concordance, whereas the minimum value −1 is obtained either in the case of unanimous discordance, or if there exists a strong veto situation on at least one criterion. The median situation 0 represents a case of undeterminateness: either the arguments in favour of an outranking are compensated by those against it or, a positive concordance in favour of the outranking is outbalanced by a potential (weak) veto situation.

It is now easy to recover the semantics linked to this bipolar-valued characterisation from our earlier considerations (see Subsection 2.1). For any two alternatives x and y of X,

- $\widetilde{S}(x,y) = +1$ means that assertion "xSy" is *clearly validated.*
- $\widetilde{S}(x,y) > 0$ means that assertion "xSy" is *more validated than non-validated.*
- $\widetilde{S}(x,y) = 0$ means that assertion "xSy" is *undetermined.*
- $\widetilde{S}(x,y) < 0$ means that assertion "xSy" is *more non-validated than validated.*
- $\widetilde{S}(x,y) = -1$ means that assertion "xSy" is *clearly non-validated.*

Definition 2.3. *The set X associated to a bipolar-valued characterisation $\widetilde{S}$ of the outranking relation $S \in X \times X$ is called a* bipolar-valued outranking digraph*, denoted $\widetilde{G}(X, \widetilde{S})$.*

The crisp outranking relation S can be constructed via its bipolar-valued characterisation. S is the set of pairs (x, y) of $X \times X$ such that $\widetilde{S}(x,y) > 0$. We write $G(X,S)$ the corresponding so-called *crisp outranking digraph* associated to $\widetilde{G}(X, \widetilde{S})$.

Example 1 *Consider the set $X_1 = \{a, b, c, d, e\}$ of alternatives evaluated on a coherent family $F_1 = \{1, \ldots, 5\}$ of criteria of equal weights (see left part of Table 2.1). To each criterion is associated a rational preference scale in $[0, 1]$ and an indifference threshold of 0.1, a preference threshold of 0.2, a weak veto threshold of 0.6, and a strong veto threshold of 0.8.*

Based on the performances of the five alternatives on the criteria, we compute the bipolar-valued outranking relation $\widetilde{S}_1$ shown in the right part of Table 2.1. The crisp outranking digraph $G_1(X_1, S_1)$ associated to the bipolar-valued outranking digraph $\widetilde{G}_1(X_1, \widetilde{S}_1)$ is shown in Figure 2.1. Note the dotted arc from alternative e to d which represents an undetermined outranking. This situation is not expressible in a standard Boolean-valued characterisation of the outranking. Consequently, the ('positive') negation of the general $\widetilde{S}$ relation is not identical to the complement of S in $X \times X$.

The reader, familiar with the Electre methodology, may have noticed much ressemblance between the bipolar-valued characterisation $\widetilde{S}$ and the classical Electre-type valuations of an outranking relation. It is important to notice, however, that the latter do not necessarily respect the semantics of the bipolar credibility calculus. In particular, the 1/2 value does in general

alternatives	coherent family of criteria 1	2	3	4	5	$\widetilde{S}_1$ a	b	c	d	e
a	0.52	0.82	0.07	1.00	0.04	1.0	-0.2	-1.0	0.6	0.4
b	0.96	0.27	0.43	0.83	0.32	0.4	1.0	0.2	0.2	0.4
c	0.85	0.31	0.61	0.41	0.98	0.2	0.4	1.0	0.4	0.6
d	0.30	0.60	0.74	0.02	0.02	-1.0	-1.0	-1.0	1.0	-1.0
e	0.18	0.11	0.23	0.94	0.63	0.2	0.2	-0.4	0.0	1.0

Table 2.1. Example 1: Random performance table and bipolar-valued outranking relation

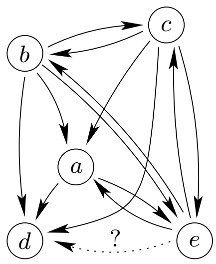

Fig. 2.1. Example 1: Associated crisp digraph and undetermined arc

not have the meaning of undetermined validation which is given here to the 0 credibility degree. Furthermore, the bipolar valuation of the outranking relation is solely based on sums and differences of weights of individual criteria.

Let us now introduce some further concepts which are used in this article. The *order* n of the digraph $\widetilde{G}(X, \widetilde{S})$ is given by the cardinality of X, whereas the *size* p of $\widetilde{G}$ is given by the cardinality of S. A *path of order* $m \leqslant n$ in $\widetilde{G}(X, \widetilde{S})$ is a sequence $(x_i)_{i=1}^m$ of alternatives of X such that $\widetilde{S}(x_i, x_{i+1}) \geq 0$, $\forall i \in \{1, \ldots, m-1\}$. A *circuit of order* $m \leqslant n$ is a path of order m such that $\widetilde{S}(x_m, x_1) \geq 0$.

Definition 2.4. *An* odd chordless circuit $(x_i)_{i=1}^m$ *is a circuit of odd order* m *such that* $\widetilde{S}(x_i, x_{i+1}) \geq 0$, $\forall i \in \{1, \ldots, m-1\}$, $\widetilde{S}(x_m, x_1) \geq 0$ *and* $\widetilde{S}(x_i, x_j) < 0$ *otherwise.*

Following a result by Bouyssou (2006) which extends the results of Bouyssou (1996) to the bipolar-valued case, it appears that, apart from certainly being reflexive, the bipolar-valued outranking digraphs do not necessarily have any particular relational properties such as transitivity or total comparability. Indeed he shows that, with a sufficient number of criteria, it is always possible to define a performance table such that the associated crisp outranking digraph renders any given reflexive binary relation. This rather positive result from a methodological point of view, namely that the outranking based methodology is universal, bears however a negative algorithmic consequence. Indeed, as we will show in Sections 3 and 4, solving the choice problem based on a bipolar-valued outranking relation is a non-trivial algorithmic problem in case of non-transitive and partial outrankings.

Before switching to the following subsection, it is important to underline here that the starting point of this study is deliberately a *given* performance table, a set of threshold and veto functions as well as significance weights which are all clearly defined and have been acknowledged by the DM[3].

Historically, in the context of outranking relations, the progressive choice problem has been solved by using the independent outranking set, i.e. the kernel of a digraph (Roy, 1968, 1985). Let us now define this concept in a bipolar-valued outranking digraph.

2.3 On kernels in bipolar-valued outranking digraphs

Definition 2.5. *Let Y be a non-empty subset of X.*

1. *Y is said to be* outranking *(resp.* outranked*) in $\widetilde{G}(X,\widetilde{S})$ if and only if $x \notin Y \;\Rightarrow\; \exists y \in Y : \widetilde{S}(y,x) > 0$ (resp. $\widetilde{S}(x,y) > 0$).*
2. *Y is said to be* independent *(resp.* strictly independent*) in $\widetilde{G}(X,\widetilde{S})$ if and only if for all $x \neq y$ in Y we have $\widetilde{S}(x,y) \leqslant 0$ (resp. $\widetilde{S}(x,y) < 0$).*
3. *Y is called an* outranking *(resp.* outranked*)* kernel *if and only if it is an outranking (resp. outranked) and independent set.*
4. *Y is called a* determined *outranking (resp. outranked)* kernel *if and only if it is an outranking (resp. outranked) and strictly independent set.*

Example 1 (continued) *In the crisp digraph G_1 (see Figure 2.1) we can observe two determined outranking kernels, namely the singletons $\{b\}$ and $\{c\}$. The digraph also contains one outranked kernel, namely the pair $\{d,e\}$. Note that alternatives d and e are independent (but not strictly independent) from each other.*

The set Y may be characterised with the help of bipolar-valued membership assertions $\widetilde{Y} : X \to \mathcal{L}$, denoting the credibility of the fact that $x \in Y$ or not, for all $x \in X$. The set $\widetilde{Y}$ is called a bipolar-valued characterisation of Y, or for short a *bipolar-valued set* in $\widetilde{G}(X,\widetilde{S})$. The semantics linked to this characterisation can again be derived from the properties of the bipolar-valued scale $\mathcal{L}$ (see Subsection 2.1).

In the following paragraphs, we recall useful results from Bisdorff et al. (2006). They allow us to establish a link between the classical graph theoretic and algebraic representations of kernels (via their bipolar-valued characterisations).

[3] Tackling impreciseness issues in these data is out of the scope of this paper. For first attempts to cope with this topic in a bipolar-valued credibility calculus framework, see (Bisdorff, 2004).

Proposition 2.6. *The outranking (resp. outranked) kernels of $\widetilde{G}(X, \widetilde{S})$ are among the bipolar-valued sets $\widetilde{Y}$ satisfying the respective following bipolar-valued kernel equation systems:*

$$\max_{y \neq x}[\min(\widetilde{Y}(y), \widetilde{S}(y,x))] = -\widetilde{Y}(x), \quad \textit{for all } x \in X; \tag{2.3}$$

$$\max_{y \neq x}[\min(\widetilde{S}(x,y), \widetilde{Y}(y))] = -\widetilde{Y}(x), \quad \textit{for all } x \in X. \tag{2.4}$$

Let $\mathcal{Y}^+$ and $\mathcal{Y}^-$ denote the set of bipolar-valued sets verifying the respective kernel equation systems (2.3) and (2.4) above. Let $\widetilde{Y}_1$ and $\widetilde{Y}_2$ be two elements of $\mathcal{Y}^+$ (or $\mathcal{Y}^-$). $\widetilde{Y}_1$ is said to be *at least as sharp as* $\widetilde{Y}_2$ (denoted $\widetilde{Y}_2 \preceq \widetilde{Y}_1$) if and only if for all x in X either $\widetilde{Y}_1(x) \leqslant \widetilde{Y}_2(x) \leqslant 0$ or $0 \leqslant \widetilde{Y}_2(x) \leqslant \widetilde{Y}_1(x)$. The $\preceq$ relation defines a partial order (antisymmetrical and transitive) (Bisdorff, 1997). If $\widetilde{Y}(x) \neq 0$ for each x in X, $\widetilde{Y}$ is called a *determined* bipolar-valued set.

Theorem 2.7 (Bisdorff, Pirlot, Roubens, 2006).

1. *There exists a one-to-one correspondence between the maximal sharp determined sets in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$) and the determined outranking (resp. outranked) kernels in $\widetilde{G}$.*
2. *Each maximal sharp set in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$) characterises an outranking (resp. outranked) kernel in $\widetilde{G}$.*

Proof. The first result, specialised to determined sets, is proved in Bisdorff et al. (2006, Theorem 1). The second one results directly from the kernel equation systems of Proposition 2.6. □

The maximal sharp sets in $\mathcal{Y}^+$ (resp. $\mathcal{Y}^-$) deliver thus *outranking* (resp. *outranked*) kernel characterisations. Let us furthermore underline, that it may also happen that neither $\mathcal{Y}^+$ nor $\mathcal{Y}^-$ contain any determined, or even partially determined sets at all. Such a case arises for example if $\widetilde{G}$ is an odd chordless circuit (see Definition 2.4).

Example 1 (continued) *Recall that the crisp outranking digraph G_1 contains two outranking kernels and one outranked kernel. The bipolar-valued characterisations of these kernels are shown in Table 2.2. The outranking kernel $\{c\}$ is more determined than $\{b\}$ and is therefore the more credible instance. Indeed, one can easily verify that the degrees of logical determination of the membership assertions for $\{c\}$ are higher than those for $\{b\}$ (see Definition 2.2). Concerning the outranked kernel $\{d, e\}$, it is worthwhile noting that alternative d belongs to it with certainty, whereas the belonging of alternative e to this kernel depends on the undetermined situation dSe. In the context of a progressive method, if this latter outranking becomes more*

$\widetilde{Y}$	a	b	c	d	e
$\{b\}$	-0.2	**0.2**	-0.2	-0.2	-0.2
$\{\mathbf{c}\}$	-0.2	-0.4	**0.4**	-0.4	-0.4
$\{d, e\}$	-0.6	-0.2	-0.4	**1.0**	**0.0**

Table 2.2. Example 1: bipolar-valued characterisations of the kernels

true than false at a later stage, then e can be dropped from the kernel without any regret. On the opposite, if the outranking becomes more non-validated than validated, then e remains part of the then determined kernel $\{d, e\}$.

In the past (see e.g. (Bisdorff and Roubens, 2003)), the authors have promoted the *most determined* outranking kernel in a bipolar-valued outranking digraph $\widetilde{G}$ as a convenient choice recommendation in a progressive choice problem.

Example 1 (continued) *The reader can indeed easily verify in the performance table of Example 1 (see Table 2.1) that alternative c is performing better than alternative b. Alternative a has very contrasted performances and d indisputably presents the worst performances.*

However, recent well founded criticisms against the capacity of the outranking kernel concept to generate in general outranking digraphs a satisfactory and convincing choice recommendation led us to reconsider our method. We therefore revisit in the next section the pragmatic foundations of a progressive choice decision aiding methodology.

3 Foundations of the Rubis choice decision aiding methodology

First, we revisit the choice problematique in order to identify the type of pragmatic decision aiding we are interested in. A brief comparison with the classical Electre methodology will underline similarities and differences with the Rubis one. Finally, we present new foundations for the choice decision aiding methodology.

3.1 The choice problematique

From a *classical* OR point of view, the choice problem is the search for one best or optimal alternative. From a *decision aiding* point of view, however, the assistance we may offer the DM depends on the nature of the decision aiding process we support.

Choice and elimination:
Following the tradition (Roy, 1985; Roy and Bouyssou, 1993), we call *choice problematique* the category of decision problems consisting of the search for a single best alternative. Symmetrically to this choice problematique, we define the *elimination* problematique as the category of decision problems, whose objective it is to search for a single *worst* alternative.

The interest of considering both opposite problematiques will appear later in Subsections 3.3 and 4.3, where we show that, due to the intransitivity of the outranking relation, certain sets of alternatives can be considered a potential choice, as well as, a potential elimination recommendation (which makes the recommendation ambiguous in both problematiques).

Following the symmetric design of the bipolar credibility calculus, both the choice, and the elimination problematique can be tackled similarly. As the first one is much more common, we will in the sequel exclusively focus on the choice problematique.

Type of decision aiding process:
A *decision aiding method* is a particular way to solve a given decision problem. A *choice recommendation* is the output of such a decision aiding method in the particular context of the choice problematique.

Following the nature of the decision aiding process, we have to distinguish between two general kinds of choice problems. On the one hand, choice problems which require the single best alternative to be uncovered in a single decision aiding step, and, on the other hand, choice problems which allow to progressively uncover the single best alternative through the implementation of an iterative, progressive multiple step decision aiding process.

In the first case, a choice recommendation must always propose a single best alternative, whereas in the second case, the choice recommendation is a provisional advice that should, given the current available information, propose all plausible candidates for a final solution. It is in fact a set of potentially best alternatives which has to be refined via further interactions with the DM. We have here to clearly distinguish between a *current* and the *eventual* choice recommendation consisting ideally of the single best alternative. If not, this last recommendation requires to be further analysed by the DM himself, in view of determining his ultimate choice.

We focus in the sequel on the resolution of this progressive decision aiding problem, in the tradition of the classical ELECTRE methods.

3.2 The ELECTRE choice decision aiding method

The progressive choice problem is extensively discussed and promoted in the context of multiple criteria decision aiding in Roy and Bouyssou (1993) who explain that it is important that the non-retained alternatives for the current choice recommendation are left out for well-founded reasons, acknowledged and approved by the DM. Instead of forcing the decision aiding procedure to elicit a single best alternative at any cost, it is indeed preferable to obtain a set Y of potential candidates for the choice, as long as it can be plainly justified on the basis of the currently available preferential information.

Starting from this methodological position, Roy defines two principles for the construction of a choice recommendation. Subset Y of X is a choice recommendation if:

1. Each alternative which in not selected in Y is outranked by at least one alternative of Y;
2. The number of retained alternatives in the set Y is as small as possible.

The first principle counterbalances the second one. Indeed, it tends to keep the cardinality of the choice recommendation high enough to guarantee that no potentially best alternative is missed out. The second principle tends to keep its cardinality as small as possible in order to focus on the single best choice.

In the context of the Electre methods, Roy (1968, 1985) proposes to use as provisional choice recommendation the concept of *outranking kernel*, i.e., an independent and outranking set. One can indeed easily check that this recommendation verifies both principles. According to Roy, a choice recommendation has furthermore to be unique. The existence of a unique outranking kernel is, however, only guaranteed when the digraph does not contain any circuits at all (Berge, 1970). To avoid a possible emptiness or multiplicity of outranking kernels, Roy (1968) initially proposed in the Electre I method to consider the alternatives belonging to maximal circuits as ties. The reduction of the outranking digraph along these ties results in a digraph that always admits a unique outranking kernel. The alternatives gathered in such a maximal circuit might, however, not be all equivalent and behave differently when compared to alternatives exterior to the circuit. Furthermore, the validation of the arcs of such a circuit may be problematic due to imprecision in the data or the preferential information provided by the DM. All in all, these difficulties in the clear interpretation of those circuits led to the development of the Electre IS method (see Roy and Bouyssou, 1993). There, robustness considerations allow to remove certain arcs of the outranking digraph leading to a circuit-free graph containing a unique outranking kernel. Note finally that in both methods, the outranking relation is not viewed on a valued credibility scale. The double requirement of sufficient concordance and absence of vetoes is used instead for a crisp validation of pairwise outranking situations.

In this work we do not follow the same approach, even if the bipolar-valued framework would allow it. We instead revisit the very foundations of a progressive choice decision aiding methodology in order to discover how the bipolar-valued concept of outranking kernel may deliver a satisfactory choice recommendation without having to express doubts about a given bipolar-valued characterisation of the outranking relation.

3.3 New foundations for a progressive choice decision aiding methodology

In this subsection we introduce five principles (two from the previous discussion and three new ones) that the construction of a choice recommendation in a progressive decision aiding method should follow.

$\mathcal{P}_1$: **Non-retainment for well motivated reasons**
Each non-retained alternative must be put aside for well motivated reasons in order to avoid to miss any potentially best alternative.

A similar formulation is that each non-retained alternative must be considered as worse as at least one alternative of the choice recommendation.

$\mathcal{P}_2$: **Minimal size**
The number of alternatives retained in a choice recommendation should be as small as possible.

This requirement is obvious when recalling that the goal of the choice problem is to find a single best alternative and that ultimately, a choice recommendation containing a single element concludes the progressive decision aiding process.

$\mathcal{P}_3$: **Efficient and informative refinement**
Each step of the progressive decision aiding must deliver an *efficient and informative refinement* of the previous recommendation.

The currently delivered recommendation should focus on new and previously unknown preference statements, such that the progressive decision aiding process can converge to a single choice recommendation as quickly and efficiently as possible. Note that a progressive decision aiding process is not required to go on until a single best alternative can be recommended. As already mentioned, it may be up to the DM to determine the ultimate choice from the eventual recommendation of the decision aiding.

Principle $\mathcal{P}_3$ is quite similar to the previous principle and appears to make it redundant. In the following section, however, when implementing the RUBIS method, their strategic difference will become apparent.

$\mathcal{P}_4$: **Effective recommendation**
The recommendation should not correspond simultaneously to a *choice* and an *elimination* recommendation.

This principle avoids the formulation of ambiguous recommendations, i.e. both outranking and outranked sets of alternatives, which could appear in intransitive and partial outranking relations. It is worthwhile noting that in a situation where all decision alternatives are either considered to be pairwisely equivalent or incomparable, no effective choice recommendation can be made.

$\mathcal{P}_5$: **Maximal credibility**
The choice recommendation must be as credible as possible with respect to the preferential knowledge available in the current step of the decision aiding process.

As the credibility degrees in the bipolar-valued outranking digraph represent the more or less overall concordance or consensus of the criteria to support an outranking situation, it seems quite natural that in the case of several potential choice recommendations, we recommend the one(s) with the highest determinateness of the membership assertions.

As mentioned before, the first two principles are identical to those proposed by Roy (see Subsection 3.2). However, alone they are not sufficient to generate satisfactory choice recommendations. The three additional principles $\mathcal{P}_3$, $\mathcal{P}_4$, and $\mathcal{P}_5$ will show their operational value when translated in Section 4 into properties in the bipolar-valued outranking digraph.

Definition 3.1. *A choice recommendation which verifies the five principles above is called a* RUBIS choice recommendation (RCR).

Our goal in the following section is to determine which graph theory-related object these properties characterise as a convincing choice recommendation.

4 Tackling the choice problem in the bipolar-valued outranking digraph

Let us note beforehand that obvious RUBIS choice recommendations exist in case the outranking relation is transitive, namely all maximal alternative(s). However, as already mentioned earlier, the crisp outranking digraphs that we obtain from the bipolar-valued characterisation of an outranking relation are in general not transitive. This clearly motivates the necessity to find a procedure which computes a choice recommendation verifying the five principles for any possible reflexive binary relation.

Throughout this section, we illustrate our discourse via the following didactic example[4].

[4] B.Roy, 2005, private communication.

$\widetilde{S}_2$	a	b	c	d	e
a	1.0	0.2	-1.0	-0.7	-0.8
b	-0.6	1.0	0.8	1.0	0.0
c	-1.0	-1.0	1.0	0.2	0.8
d	0.6	-0.6	-1.0	1.0	-0.4
e	-1.0	-0.8	-0.4	-0.6	1.0

Table 4.1. Example 2: the bipolar-valued outranking relation

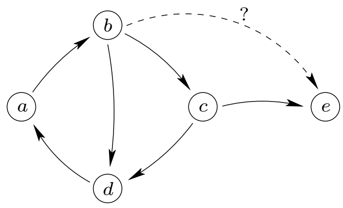

Fig. 4.1. Example 2: the associated crisp digraph and an undetermined arc

Example 2 *Let $\widetilde{G}_2(X_2, \widetilde{S}_2)$ be a bipolar-valued outranking digraph, where $X_2 = \{a, b, c, d, e\}$ and $\widetilde{S}_2$ is given in table 4.1 and the associated crisp digraph $G_2(X_2, S_2)$ is represented in figure 4.1.*

Let us now analyse the previously mentioned principles one by one and present their translations in terms of the concepts presented in Section 2. Note that all the directed concepts linked to an outranking property can symmetrically be reused in an elimination problematique via the corresponding outranked properties.

4.1 Non-retainment for well motivated reasons (principle: $\mathcal{P}_1$)

In terms of the bipolar-valued outranking relation, principle $\mathcal{P}_1$ amounts to saying that each non-retained alternative should be outranked by at least one alternative of the choice recommendation.

$\mathcal{R}_1$: **Outranking**
An RCR[5] is an outranking set in $\widetilde{G}(X, \widetilde{Y})$.

Example 2 (continued) *The sets $\{a, b, e\}$, $\{b, c, d\}$, as well as $\{a, b, c\}$ for instance, are all outranking sets.*

[5] RUBIS choice recommendation (see Definition 3.1).

4.2 Minimal size and efficient and informative refinement (principles: $\mathcal{P}_2$ and $\mathcal{P}_3$)

In this subsection we show that these two principles are closely linked. To rewrite principle $\mathcal{P}_2$ of *minimal size* in the present context, we first need to define some concepts related to graph theory.

Definition 4.1.

1. *The* outranking neighbourhood $\Gamma^+(x)$ *of a node (or equivalently an alternative) x of X is the union of x and the set of alternatives which are outranked by x.*
2. *The* outranking neighbourhood $\Gamma^+(Y)$ *of a set Y is the union of the outranking neighbourhoods of the alternatives of Y.*
3. *The* private outranking neighbourhood $\Gamma_Y^+(x)$ *of an alternative x in a set Y is the set $\Gamma^+(x) \setminus \Gamma^+(Y \setminus \{x\})$.*

For a given alternative x of a set Y, the set $\Gamma_Y^+(x)$ represents the *individual* contribution of x to the outranking quality of Y. If the private outranking neighbourhood of x in Y is empty, this means that, when x is dropped from this set, Y still remains an outranking set. From this observation one can derive the following definition.

Definition 4.2. *An outranking (resp. outranked) set Y is said to be* irredundant *if all the alternatives of Y have non-empty private outranking (resp. outranked) neighbourhoods.*

The formal counterpart of the minimal size principle is therefore that of irredundancy of the set.

Example 2 (continued) *$\{a,b,e\}$, $\{b,c,d\}$, $\{b,e,d\}$, and $\{a,c\}$ are irredundant outranking sets. $\{a,b,c\}$, listed in the context of principle $\mathcal{R}_1$, is not irredundant outranking because alternative b has an empty private neighbourhood in this set.*

Let us now switch to principle $\mathcal{P}_3$ (*efficient and informative refinement*), whose primary objective is to avoid that, in the case of a provisional choice recommendation, the DM may notice a best sub-choice without any further analyses. We require therefore that a choice recommendation Y should be such that the digraph restricted to the nodes of Y does not contain any obvious sub-choice recommendation. Consequently, at each stage of the decision aiding process, the provisional choice recommendation must focus on new and previously undetermined or unknown preference statements. Let us illustrate this with a short example.

Example 3 *Consider the problem shown on the crisp digraph represented in figure 4.2. Both highlighted sets* $Y_1 = \{a, b\}$ *and* $Y_2 = \{a,\ d, e, f,\ \ldots, z\}$ *verify the principles* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *as outranking irredundant sets. One would be tempted to prefer* Y_1 *to* Y_2 *because of its lower cardinality. Nevertheless,* Y_1 *contains information which is already confirmed at this stage of the progressive search, namely that the statement "a outranks b" is validated. In the case of the choice* Y_2*, the next step of the search will focus on alternatives which presently are incomparable. If a further analysis step would focus on the set* Y_1*, then it is quite difficult to imagine that the DM will be able to forget about the already confirmed validation of the statement "a outranks b". He will most certainly consider a as the choice, which might however not be the best decision alternative, as a is not outranking any of the alternatives of* $\{d, e, f, \ldots, z\}$.

According to principle $\mathcal{P}_3$ *we therefore recommend* Y_2 *as a choice recommendation.*

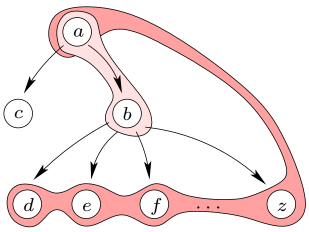

Fig. 4.2. Example 3: An unstable ($\{a, b\}$) and a stable ($\{a, d, e, f, \ldots, z\}$) set.

In view of the previous considerations and the output generated by principles $\mathcal{P}_1$ and $\mathcal{P}_2$, it is quite natural to define the concept of stability as follows:

Definition 4.3. *An outranking (resp. outranked) set* Y *in* $\widetilde{G}(X, \widetilde{Y})$ *is said to be stable if and only if the induced subgraph* $\widetilde{G}_Y(Y, \widetilde{S}|_Y)$ *does not contain any irredundant outranking (resp. outranked) subset.*

The outranking (resp. outranked) kernels (see Definition 2.5) of an outranking digraph verify this property of stability. Nevertheless, as already mentioned in Subsection 3.2 and as it is shown in the following property, the existence of an outranking (resp. outranked) kernel is not guaranteed in an outranking digraph.

Property 4.4. If a digraph $\widetilde{G}(X, \widetilde{S})$ has no outranking (resp. outranked) kernel, it contains a chordless circuit of odd order.

Proof. This property represents the contraposition of Richardson's general result: If a digraph contains no chordless circuit of odd order, then it has an outranking (resp. outranked) kernel (see Richardson, 1953). □

The outranking kernel gives indeed a potential choice recommendation in case the outranking digraph does not contain any chordless circuit of odd order. Consider now the case where a potential choice recommendation, resulting from principles $\mathcal{P}_1$ and $\mathcal{P}_2$, consists in a chordless circuit $Y = \{a, b, c\}$ of order 3 such that aSb, bSc and cSa. Such a choice recommendation is clearly neither a kernel nor is it a stable recommendation. Nevertheless, it may be an interesting provisional recommendation because it presents three alternatives to the DM which do not contain obvious information on the possible single choice at this step of the progressive search. In fact, a, b and c can be considered as equivalent potential candidates for the choice in the current stage of the decision process.

Neither the concepts of stability and irredundancy nor that of outranking kernel are in fact sufficient for guiding the search for a choice recommendation in a general outranking digraph. In the first case, potentially interesting choice recommendations are left out and in the latter case, nothing guarantees the existence of a kernel in an outranking digraph. In order to overcome these difficulties, we introduce the concept of *hyperindependence*, an extension of the independence property discussed in Subsection 2.3.

Definition 4.5. *A set Y is said to be (strictly)* hyperindependent *in $\widetilde{G}$ if it consists of chordless circuits of odd order $p \geq 1$ which are (strictly) independent of each other.*

Note that in Definition 4.5 above, singletons are assimilated to chordless circuits of (odd) order 1. Principles $\mathcal{P}_2$ and $\mathcal{P}_3$ can now be translated into the following formal property:

$\mathcal{R}_2$: **Hyperindependence**
An RCR is a hyperindependent set in $\widetilde{G}(X, \widetilde{Y})$.

As a direct consequence, we can define the concept of *hyperkernel*.

Definition 4.6. *A hyperindependent (resp. strictly hyperindependent) outranking (resp. outranked) set is called an outranking (resp. outranked)* hyperkernel *(resp.* determined hyperkernel*).*

Example 2 (continued) *Set $\{a, b, d, e\}$ (see Figure 4.1) is an outranking hyperkernel. The undetermined outranking relation between b and e implies that the set is not strictly hyperindependent. Note here that this obvious potential choice recommendation would have been left out if the search was restricted to outranking kernels.*

In case the outranking digraph does not contain any chordless circuits of odd order 3 and more, the outranking kernels of the digraph deliver potential choice recommendations verifying the two first Rubis principles. In the general case however, the RCR will consist of at least one outranking hyperkernel of the digraph.

4.3 Effective and maximally credible recommendation (principles $\mathcal{P}_4$ and $\mathcal{P}_5$)

In order to translate principle $\mathcal{P}_4$ (*effective recommendation*), we introduce the concept of *strict outranking* set. Recall that one can associate an outranking (resp. outranked) set Y with a bipolar-valued characterisation $\widetilde{Y}^+$ (resp. $\widetilde{Y}^-$). It may happen that both kernel characterisations are solutions of the respective kernel equation systems of Proposition 2.6. In order to determine in this case whether Y is in fact an outranking or an outranked set, it is necessary to specify which of its bipolar-valued characterisations is the more determined.

We extend therefore the concept of determinateness of propositional statements (see Definition 2.2) to bipolar-valued characterisations of sets.

Definition 4.7. *The determinateness $D(\widetilde{Y})$ of the bipolar-valued characterisation $\widetilde{Y}$ of a set Y is given by the average value of the determinateness degrees $D(\widetilde{Y}(x))$ for all x in X.*

In view of the bipolar definition of the global outranking and concordance indexes (Formulae 2.1 and 2.2), which solely balance rational significance weights, we define here the overall determinateness of a bipolar-valued set characterisation as the mean of all the individual membership determinatenesses. Nevertheless other aggregation operators could be used as well.

We can now define the concept of strictness as follows:

Definition 4.8.

1. *A set Y which is outranking and outranked with the same determinateness, i.e., $D(\widetilde{Y}^+) = D(\widetilde{Y}^-)$ is called a* null set.
2. *A set Y for which $D(\widetilde{Y}^+) > D(\widetilde{Y}^-)$ (resp. $D(\widetilde{Y}^-) > D(\widetilde{Y}^+)$) is called a* strict *outranking set (resp. outranked set).*

One can now translate the principle of effectiveness $\mathcal{P}_4$ into the following formal property:

$\mathcal{R}_3$: **Strict outranking**
An RCR is a strict outranking set in $\widetilde{G}(X, \widetilde{Y})$.

This concept allows to solve the problem raised by the following example.

Example 4 *Consider the crisp outranking digraph represented on figure 4.3* [6] *(for the sake of simplicity we suppose that all the arcs which are drawn (resp. not drawn) represent a credibility of the outranking of* 1 *(resp.* -1*)).* $\{a\}$ *and* $\{c\}$ *are both irredundant outranking sets with the same maximal determinateness* 1*. However, one can easily see that alternative a compares differently with b than c does. Set* $\{c\}$ *is clearly a null set. If we now require the three properties* $\mathcal{R}_1$*,* $\mathcal{R}_2$ *and* $\mathcal{R}_3$ *to be verified, only the set* $\{a\}$ *can be retained as a potential choice recommendation.*

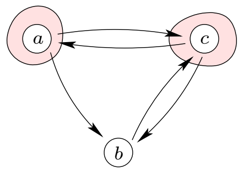

Fig. 4.3. Example 4: illustration of the necessity of Property $\mathcal{R}_3$

An immediate consequence of the effectiveness principle is that a bipolar-valued outranking digraph, which is completely symmetrical, i.e., with equal credibility degrees for all xSy and ySx, does not admit any RCR. Every outranking set will automatically be a null set. Indeed, without any asymmetrical preferential statements, it is impossible to derive any preferential discriminations that would support a convincing choice recommendation.

Finally, principle $\mathcal{P}_5$ (*maximal credibility*) involves again the idea of determinateness of bipolar-valued sets (see Definition 4.7). In the case of multiple potential choice recommendations, we recommend the most determined one, i.e., the one with the highest determinateness. Let $\widetilde{\mathcal{Y}}$ be the set of sets verifying $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ in $\widetilde{G}(Y,\widetilde{S})$.

$\mathcal{R}_4$: **Maximal determinateness**
An RCR is a choice in $\widetilde{G}(X,\widetilde{S})$ that belongs to the set

$$\widetilde{\mathcal{Y}}^* = \big\{\widetilde{Y}' \in \widetilde{\mathcal{Y}} | D(\widetilde{Y}') = \max_{\widetilde{Y} \in \widetilde{\mathcal{Y}}} D(\widetilde{Y})\big\}. \qquad (4.1)$$

Example 1 (continued) *Recall that in this example (see Subsection 2.2), we determined two outranking kernels which were potential choice recommendations (see Table 4.2). The determinateness of the kernel* $\{c\}$ (0.36) *is significantly higher than that of kernel* $\{b\}$ (0.20)*. Following property* $\mathcal{R}_4$*, we recommend in this case the first solution, namely kernel* $\{c\}$*.*

[6] Inspired from Roy and Bouyssou (1993).

$\widetilde{Y}$	a	b	**c**	d	e	$D(\widetilde{Y})$	
{**c**}	-0.2	-0.4	**0.4**	-0.4	-0.4	**0.36**	RCR
{b}	-0.2	**0.2**	-0.2	-0.2	-0.2	**0.20**	-

Table 4.2. Example 1: Illustration of the maximal credibility principle

In this section, we have presented the translation of the five Rubis principles into properties of sets of alternatives defined in the bipolar-valued outranking digraph. Detailed motivations for these principles have been given. They lead quite naturally to the new concept of outranking hyperkernel of an outranking digraph. Consequently, a maximally determined strict outranking hyperkernel, which by construction verifies all five principles, gives a Rubis *choice recommendation*.

The following section focuses on the construction of the hyperkernels and proposes a general algorithm for computing the Rubis choice recommendations in a given (non-symmetrical) bipolar-valued outranking digraph.

5 Computing the Rubis choice recommendation

We start by presenting an algorithm which allows to determine the hyperkernels of an outranking digraph before presenting some of their properties.

5.1 Determination of the hyperkernels

If $\widetilde{G}(X,\widetilde{S})$ contains chordless circuits of odd order ($\geqslant 3$), the original outranking digraph is modified into a digraph that we will call the *chordless-odd-circuits-augmented* (COCA) outranking digraph $\widetilde{G}^{\mathcal{C}}(X^{\mathcal{C}},\widetilde{S}^{\mathcal{C}})$. Intuitively, the main idea is to “hide” the problematic circuits behind new nodes which are added to the digraph in a particular way. This may appear to be a problematic perturbation of the original information. Nevertheless, as we will see later, such a transformation does not affect the original problem but only helps to find more suitable solutions.

The procedure to obtain the COCA digraph $\widetilde{G}^{\mathcal{C}}$ is iterative. The initial digraph is written $\widetilde{G}_0(X_0,\widetilde{S}_0)$, and is equal to $\widetilde{G}(X,\widetilde{S})$. At step i, the set of nodes becomes $X_i = X_{i-1} \cup \mathcal{C}_i$, where $\mathcal{C}_i$ is a set of nodes representing the chordless circuits of odd order of $\widetilde{G}_{i-1}(X_{i-1},\widetilde{S}_{i-1})$. These nodes are called *hypernodes*. The outranking relation $\widetilde{S}_{i-1}$ is augmented by links between the nodes from X_{i-1} and those from $\mathcal{C}_i$ in the following way (the resulting relation is written $\widetilde{S}_i$)[7]:

$$\forall C_k \in \mathcal{C}_i \begin{cases} \widetilde{S}_i(C_k, x) = \bigcup_{y \in C_k} \widetilde{S}_{i-1}(y, x) & \forall x \in X_{i-1} \setminus C_k \ , \\ \widetilde{S}_i(C_k, x) = +1 & \forall x \in C_k \ , \end{cases} \tag{5.1}$$

$$\forall x \in X_{i-1}, C_k \in \mathcal{C}_i \begin{cases} \widetilde{S}_i(x, C_k) = \bigcup_{y \in C_k} \widetilde{S}_{i-1}(x, y) & \text{if } x \notin C_k \ , \\ \widetilde{S}_i(x, C_k) = +1 & \text{if } x \in C_k \ . \end{cases} \tag{5.2}$$

The iteration is stopped at step r for which $|X_r| = |X_{r+1}|$. We then define $\widetilde{G}^{\mathcal{C}}(X^{\mathcal{C}}, \widetilde{S}^{\mathcal{C}})$ as the digraph $\widetilde{G}_r(X_r, \widetilde{S}_r)$. As the order of the original digraph $\widetilde{G}$ is finite, the number of circuits it may contain is also finite. Therefore, the iteration is a finite process. Note that this iterative approach is necessary because of the fact that new chordless circuits of odd order may appear when new hypernodes are added to the digraph.

The outranking (resp. outranked) hyperkernels of $\widetilde{G}(X, \widetilde{S})$ are then determined by searching the classical outranking (resp. outranked) kernels of $\widetilde{G}^{\mathcal{C}}(X^{\mathcal{C}}, \widetilde{S}^{\mathcal{C}})$ (Bisdorff, 1997, 2006a).

5.2 Properties of the COCA outranking digraph

This extension of the digraph has two very important properties.

Property 5.1. The outranking (resp. outranked) kernels of $\widetilde{G}(X, \widetilde{S})$ are also the outranking (resp. outranked) kernels of $\widetilde{G}^{\mathcal{C}}(X^{\mathcal{C}}, \widetilde{S}^{\mathcal{C}})$.

Proof. Let us suppose that $\widetilde{G}$ contains at least one odd chordless circuit. Let Y be an outranking kernel of $\widetilde{G}$ (the case of the outranked kernels can be treated similarly). We must prove that Y is also an outranking kernel of $\widetilde{G}^{\mathcal{C}}$.

First, the elements of Y are independent in $\widetilde{G}$ and $\widetilde{G}^{\mathcal{C}}$ because no relation is added between elements of X in $X^{\mathcal{C}}$. Secondly, as Y is an outranking set in $\widetilde{G}$, each element of $X \setminus Y$ is outranked by at least one element of Y. In particular, if C_k is an odd chordless circuit of X, each node of C_k is also outranked by at least one element of Y (in X). Due to the special way $\widetilde{S}^{\mathcal{C}}$ is built, the node representing C_k in $X^{\mathcal{C}}$ is also also outranked by at least one element of Y. □

Property 5.2. The digraph $\widetilde{G}^{\mathcal{C}}(X^{\mathcal{C}}, \widetilde{S}^{\mathcal{C}})$ contains at least one outranking (resp. outranked) hyperkernel.

[7] For the sake of simplicity, an element C_k of $\mathcal{C}_i$ will represent a node of X_i as well as a the set of nodes of X_{i-1} representing the circuit C_k.

Proof. Following from the construction principle of the COCA digraph (see Equations 5.1 and 5.2), a hypernode *inherits* the outranking (outranked) characteristics of its corresponding odd chordless circuit. A direct consequence of this inheritance is that the outranking, as well as the outranked, neighbourhood of the odd chordless circuit are inherited by the hypernode. Furthermore, the individual nodes of each odd chordless circuit are outranked by and are outranking the hypernode with a credibility of +1 (indifference).

Let us now suppose that $\widetilde{G}(X, \widetilde{S})$ contains no outranking kernel (a similar proof can be given for the outranked kernels). According to Property 4.4 this means that $\widetilde{G}(X, \widetilde{S})$ contains at least one odd chordless circuit. One can easily understand that if the structure of the digraph requires an element $x \in X$ of an odd chordless circuit C_k to be in an irredundant outranking set Y, due to the odd number of elements of that particular circuit, one of the two direct neighbours of x in the circuit will also be added to Y. Consequently, Y cannot be kernel in that situation.

Due to the particular construction of the associated COCA digraph $\widetilde{G}^{\mathcal{C}}$, there exists, for each odd chordless circuit, a hypernode which inherits its properties, and which is considered as indifferent to it. Consequently, each element of each odd chordless circuit in $\widetilde{G}^{\mathcal{C}}$ is outranked by, and is outranking, a hypernode. Furthermore, each of the hypernodes has the same outranking and outranked neighbourhoods as its corresponding odd chordless circuit.

Finally, the element x of the odd chordless circuit C_k will no longer be problematic in the construction of the outranking kernels of $\widetilde{G}^{\mathcal{C}}$ because there exists at least one hypernode which is equivalent to x, and which inherits from the outranking neighbourhoods of C_k. Consequently C_k (as a hypernode) is added to Y instead of x. □

Finally we present and discuss the effective computing of a RUBIS choice recommendation.

5.3 The RCR algorithm

Algorithm

Input: $\widetilde{G}(X, \widetilde{S})$,

1. *Construct the associated COCA digraph* $\widetilde{G}^{\mathcal{C}}(X^{\mathcal{C}}, \widetilde{S}^{\mathcal{C}})$,
2. *Extract the sets* $\widetilde{K}^+$ *and* $\widetilde{K}^-$ *of all outranking and outranked hyperkernels from* $\widetilde{G}^{\mathcal{C}}$,
3. *Eliminate the null kernels from* $\widetilde{K}^+$,
4. *Rank the elements of* $\widetilde{K}^+$ *by decreasing logical determinateness,*

Output: *The first ranked element(s) in* $\widetilde{K}^+$.

The first step of the RCR (Rubis choice recommendation, see Definition 3.1) algorithm is by far the most difficult to achieve, as the number of odd chordless circuits in a bipolar-valued outranking digraph can be huge. To study this operational difficulty, we have compiled a sample of 1000 bipolar-valued outranking digraphs generated from performances of 20 alternatives evaluated randomly on 7 to 20 criteria with random weights distributions and random thresholds. In nearly 98% of the sample, the time to compute the COCA digraph on a standard desktop computer is less than a second. In one case, we observe an execution time of around 30 seconds (due to a high number of odd chordless circuits in the digraph).

number of odd chordless circuits	#	rel. freq. (%)	cum. freq. (%)
0	735	73.5	73.5
1	116	11.6	85.1
2	65	6.5	91.6
3	25	2.5	94.1
⋮	⋮	⋮	⋮
9	1	0.1	100.0

Table 5.1. Number of odd chordless circuits in random bipolar-valued outranking digraphs of order 20 (1000 observations).

In Table 5.1, we note that nearly 75% of the sample digraphs do not admit any odd chordless circuit at all. In 100% of the observations less than 10 hypernodes are added to the original outranking digraph.

The second step of the RCR algorithm concerns the extraction of hyperkernels from the COCA digraph. From a theoretical point of view, this step is well-known to be computationally difficult (Chvátal, 1973). However, this difficulty is directly linked to the arc-density, i.e., the relative size of the digraph. Indeed, only very sparse digraphs, showing an arc-density lower than 10% in the range of digraph orders which are relevant for the choice decision aiding problematique (10-30 alternatives), may present difficulties for the search of kernels. For the test sample of 1000 random outranking digraphs of order 20, we observe a very high mean density of 82.6% with a standard deviation of 5.7%. Consequently, determining hyperkernels is in general a task which is feasible in a very reasonable time. Indeed, the mean execution time with its standard deviation for this step of the algorithm are around a thousandth of a second on a standard desktop computer.

Finally, eliminating the null hyperkernels and sorting the strict outranking hyperkernels in decreasing order of determinateness is linear in the order of the digraph and involves no computational difficulty at all.

Let us illustrate the RCR algorithm on the second example of this paper (see Section 4).

Example 2 (continued) *The bipolar-valued outranking digraph of this example (see Figure 4.1) contains a chordless circuit of order 3, namely* $\{a, b, d\}$. *The original digraph* $\widetilde{G}_2$ *is extended to the digraph* $\widetilde{G}_2^{\mathcal{C}}$ *which contains a hyper-node representing* $\{a, b, d\}$. *The corresponding outranking digraph admits an outranking kernel* $\{a, c\}$ *and a hyperkernel* $\{\{a, b, d\}, e\}$ *which is both outranking and outranked, but not with the same degree of determinateness (see Table 5.2). The first one is significantly more determined than the second one. Consequently, the* RUBIS "choice recommendation" *is* $\{\{a, b, d\}, e\}$, *where alternative* e *is in an undetermined situation.*

$\widetilde{S}_2^{\mathcal{C}}$	a	b	c	d	e	$\{a,b,d\}$	D
a	0.1	0.2	-1.0	-0.7	-0.8	1.0	
b	-0.6	1.0	0.8	1.0	0.0	1.0	
c	-1.0	-1.0	1.0	0.2	0.8	0.2	
d	0.6	-0.6	-1.0	1.0	-0.4	1.0	
e	-1.0	-8	-0.4	-0.6	1.0	-0.6	
$\{a,b,d\}$	1.0	1.0	0.8	1.0	0.0	1.0	
$\{\{a,b,d\},e\}^+$	-0.6	-0.6	-0.6	-0.6	**0.0**	**0.6**	**0.5**
$\{a,c\}$	**0.2**	-0.2	**0.2**	-0.2	-0.2	-0.2	0.2
$\{\{a,b,d\},e\}^-$	0.0	0.0	0.0	-0.6	0.0	0.6	0.2

Table 5.2. Example 2: the associated COCA digraph with the bipolar-valued characterisations of its outranking (+) and outranked (-) hyperkernels.

Let us finish this last section by indicating that all the examples of this paper have been computed with the free Python module *digraphs* (Bisdorff, 2006b) which allows to manipulate bipolar-valued digraphs and to determine the RCR from a given performance table.

Concluding remarks

In this paper we defined new operational instruments, namely the strict outranking hyperkernel and the chordless circuits augmented digraph, which contribute to enrich the set of decision aiding tools for the choice problematique. New concepts, such as the RUBIS choice recommendation, defined on a bipolar-valued outranking digraph, adapt and extend the traditional theoretical and pragmatic framework in which the choice problem is generally tackled. Some topics remain of course untouched. In particular, the authors' future challenge will be to illustrate with a successful Rubis decision aiding practice, that a DM may indeed enhance his actual choices.

Acknowledgements: The authors would like to thank Bernard Roy and an anonymous referee for their helpful comments.

References

Belton, V. and Stewart, T. (2002). *Multiple Criteria Decision Analysis: An Integrated Approach*. Kluwer Academic, Dordrecht.

Berge, C. (1970). *Graphes et hypergraphes*. Dunod, Paris.

Bisdorff, R. (1997). On computing kernels from l-valued simple graphs. In *Proceedings of the 5th European Congress on Intelligent Techniques and Soft Computing*, volume 1, pages 97–103. EUFIT'97, Aachen.

Bisdorff, R. (2000). Logical foundation of fuzzy preferential systems with application to the electre decision aid methods. *Computers & Operations Research*, 27:673–687.

Bisdorff, R. (2002). Logical foundation of multicriteria preference aggregation. In Bouyssou, D., Jacquet-Lagrèze, E., Perny, P., Slowinski, R., Vanderpooten, D., and Vincke, P., editors, *Aiding Decisions with Multiple Criteria: Essays in Honour of Bernard Roy*, pages 379–403. Kluwer Academic Publishers, Dordrecht.

Bisdorff, R. (2004). Concordant outranking with multiple criteria of ordinal significance. *4OR, Quaterly Journal of the Belgian, French and Italian Operations Research Societies, Springer-Verlag*, 2:4:293–308.

Bisdorff, R. (2006a). On enumerating the kernels in a bipolar-valued outranking digraph. In Bouyssou, D., Roberts, F., and Tsoukiás, A., editors, *Proceedings of the DIMACS-LAMSADE workshop on Voting Theory and Preference Modelling, Paris, 25–28 October 2006*, volume 6 of *Annales du LAMSADE*, pages 1–38. Université Paris-Dauphine, CNRS, Paris.

Bisdorff, R. (2006b). The Python digraph implementation for RuBy: User Manual. University of Luxembourg, http://sma.uni.lu/bisdorff/Digraph.

Bisdorff, R., Pirlot, M., and Roubens, M. (2006). Choices and kernels in bipolar valued digraphs. *European Journal of Operational Research*, 175:155–170.

Bisdorff, R. and Roubens, M. (1998). On defining and computing fuzzy kernels from l-valued simple graphs. In Ruan, D., D'hondt, P., Govaerts, P., and Kerre, E., editors, *Intelligent Systems and Soft Computing for Nuclear Science and Industry, FLINS'96 workshop*, pages 113–123. World Scientific Publishers, Singapoure.

Bisdorff, R. and Roubens, M. (2003). Choice procedures in pairwise comparison multiple-attribute decision making methods. In Berghammer, R., Möller, B., and Struth, G., editors, *Relational and Kleene-Algebraic Methods in Computer Science: 7th International Seminar on Relational Methods in Computer Science and 2nd International Workshop on Applications of Kleene Algebra*, volume 3051 of *Lecture Notes in Computer Science*, pages 1–7, Heidelberg. Springer-Verlag.

Bouyssou, D. (1996). Outranking relations: do they have special properties? *Journal of Multi-Criteria Decision Analysis*, 5:99–111.

Bouyssou, D. (2006). Notes on bipolar outranking. Personal communication.

Chvátal, V. (1973). On the computational complexity of finding a kernel. Technical report, Report No. CRM-300, Centre de recherches mathématiques, Université de Montréal.

Kitainik, L. (1993). *Fuzzy decision procedures with binary relations: towards a unified theory*. Kluwer Academic Publisher, Boston.

Richardson, M. (1953). Solution of irreflectice relations. *Ann. Math.*, 58:573–580.

Roy, B. (1968). Classement et choix en présence de points de vue multiples (la méthode ELECTRE). *Revue française d'informatique et de recherche operationelle (RIRO)*, 2:57–75.

Roy, B. (1985). *Méthodologie multicritère d'aide à la décision*. Economica, Paris.

Roy, B. and Bouyssou, D. (1993). *Aide multicritère à la décision : Méthodes et cas*. Economica, Paris.

Roy, B. and Vanderpooten, D. (1996). The European school of MCDA: Emergence, basic features and current works. *Journal of Multi-Criteria Decision Analysis*, 5:22–38.

Windelband, W. (1884). Beiträge zur Lehre vom negativen Urteil. In *Straßburger Abhandlungen zur Philosophie: Eduard Zeller zu seinem 70. Geburtstage*, pages 165–195. Strasbourg.