# Using the Kappalab R package for capacity identification in Choquet integral based MAUT


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## Abstract

The main types of approaches to capacity identification proposed in the literature are reviewed and their advantages and inconveniences are discussed. All the reviewed methods have been implemented within the Kappalab R package. Their application is illustrated on a detailed example.



**Keywords:** Multi-criteria decision aiding, multi-attribute utility theory, Choquet integral, free software.


## 1 Introduction

Let $X := X_1 \times \cdots \times X_n$, $n \geq 2$, be a set of objects of interest, some of them possibly fictive, described by a set $N := \{1, \ldots, n\}$ of attributes. The considered attributes can be cardinal or ordinal. The aim of multi-attribute utility theory (MAUT) [13] is to model the preferences of the decision maker (DM), represented by a binary relation $\succeq$, by means of an *overall utility function* $U : X \to \mathbb{R}$ such that,

$$x \succeq y \iff U(x) \geq U(y), \qquad \forall\, x, y \in X.$$

The preference relation $\succeq$ is usually assumed to be complete and transitive. As far as the overall utility function is considered, the most frequently encountered model is the *additive value model* (see e.g. [3]). In this work, we consider the more general *transitive decomposable model* of Krantz et al. [15, 2] in which $U$ is defined, for any $x = (x_1, \ldots, x_n) \in X$, by

$$U(x) := F(u_1(x_1), \ldots, u_n(x_n)), \qquad (1)$$

where the functions $u_i : X_i \to \mathbb{R}$ are called the *utility functions* and $F : \mathbb{R}^n \to \mathbb{R}$, non-decreasing in its arguments, is sometimes called the *aggregation function*. As far as the utility functions are considered, for any $x \in X$, the quantity $u_i(x)$ can be interpreted as a measure of the satisfaction of the value $x_i$ for the DM. From now on, the term *criterion* will be used to designate the association of an attribute $i \in N$ with the corresponding utility function $u_i$. For the previous decomposable model to hold, it is necessary that the preference relation is a *weakly separable* weak order (see e.g. [3]).

The exact form of the overall utility function $U$ depends on the hypotheses on which the multi-criteria decision aiding (MCDA) problem is based. When *mutual preferential independence* (see e.g. [26]) among criteria can be assumed, it is frequent to consider that the function $F$ is additive and takes the form of a weighted sum. The decomposable model given in Eq. (1) is then equivalent to the classical additive value model. The assumption of mutual preferential independence among criteria is however rarely verified in practice. In order to be able to take interaction phenomena among criteria into account, it has been proposed to substitute a monotone set function on $N$, called *capacity* [5] or *fuzzy measure* [25], to the weight vector involved in the calculation of weighted sums. Such an approach can be regarded as taking into account not only the importance of each criterion but also the importance of each subset of criteria. A natural extension of the weighted arithmetic mean in such a context is the *Choquet integral* with respect to (w.r.t.) the defined capacity [6, 17, 16].

When using the Choquet integral as overall utility function, it is necessary, for the model in Eq. (1) to make sense, that the criteria be *commensurable*, i.e. $u_i(x) = u_j(x)$ if and only if, for the decision maker, the object $x$ is satisfied to the same extent on criteria $i$ and $j$; see e.g. [11] for a more complete discussion on commensurability.

The use of a Choquet integral as an aggregation function clearly requires the prior determination of the utility functions and of the underlying capacity. The utility functions can be determined using the extension of the MACBETH methodology [1] proposed in [16]; see also [11, 10]. This task is not trivial and can take a large percentage of the time dedicated to an MCDA problem. We will not discuss it further here as the topic of this paper is the *capacity identification problem*.

Assuming that the utility functions have been determined, the learning data from which the capacity is to be identified consists of the *initial preferences* of the decision maker: usually, a partial weak order over a (small) subset of the set $X$ of objects, a partial weak order over the set of criteria, intuitions about the importance of the criteria, etc. The precise form of these prior preferences will be discussed in Section 3.

The aim of this paper is to review the main approaches to capacity identification proposed in the literature that can be applied to Choquet integral based MAUT. As we shall see, most of the presented methods can be stated as optimization problems. They differ w.r.t. the objective function and the preferential information they require as input. For each of the reviewed methods, we point out its main advantages/disadvantages. The last part of this paper is devoted to a presentation of the discussed identification methods through the Kappalab toolbox [9]. Kappalab, which stands for "laboratory for capacities", is a package for the GNU R statistical system [23], which is a Matlab like free software environment for statistical computing and graphics. Kappalab contains high-level routines for capacity and non-additive integral manipulation on a finite setting which can be used in the framework of decision aiding or cooperative game theory. All the reviewed identification methods have been implemented within Kappalab and can be easily used through the high-level R language.

The paper is organized as follows. The second section is devoted to the presentation of the Choquet integral as an aggregation operator. In the third section, we discuss the form of the preferential information from which the capacity is to be identified. The next section contains a review of the main approaches to capacity identification existing in the literature. The application of the reviewed methods is illustrated in the last section through the use of the Kappalab R package.

In order to avoid a heavy notation, we will omit braces for singletons and pairs, e.g., by writing $\mu(i)$, $N\backslash ij$ instead of $\mu(\{i\})$, $N\backslash\{i,j\}$. Furthermore, cardinalities of subsets $S, T, \dots$, will be denoted by the corresponding lower case letters $s, t, \dots$

## 2 The Choquet integral as an aggregation operator

### 2.1 Capacities and Choquet integral

As mentioned in the introduction, *capacities* [5], also called *fuzzy measures*, can be regarded as generalizations of weighting vectors involved in the calculation of weighted sums.

**Definition 2.1** *A capacity on $N$ is a set function $\mu : 2^N \to [0,1]$ satisfying $\mu(\emptyset) = 0$, $\mu(N) = 1$, and, for any $S, T \subseteq N$, $S \subseteq T \Rightarrow \mu(S) \leq \mu(T)$.*

When using a capacity to model the importance of the subsets of criteria, a suitable aggregation operator that generalizes the weighted arithmetic mean is the Choquet integral ; see e.g. [16, 18].

**Definition 2.2** *The Choquet integral of a* *function $x : N \to \mathbb{R}$ represented by the vector $(x_1, \ldots, x_n)$ w.r.t. a capacity $\mu$ on $N$ is defined by*

$$C_\mu(x) := \sum_{i=1}^{n} x_{\sigma(i)}[\mu(A_{\sigma(i)}) - \mu(A_{\sigma(i+1)})],$$

*where $\sigma$ is a permutation on $N$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$. Also, $A_{\sigma(i)} := \{\sigma(i), \ldots, \sigma(n)\}$, for all $i \in \{1, \ldots, n\}$, and $A_{\sigma(n+1)} := \emptyset$.*

### 2.2 The Möbius transform of a capacity

Any set function $\nu : 2^N \to \mathbb{R}$ can be uniquely expressed in terms of its *Möbius representation* [24] by

$$\nu(T) = \sum_{S \subseteq T} m_\nu(S), \qquad \forall\, T \subseteq N, \tag{2}$$

where the set function $m_\nu : 2^N \to \mathbb{R}$ is called the *Möbius transform* or *Möbius representation* of $\nu$ and is given by

$$m_\nu(S) = \sum_{T \subseteq S} (-1)^{s-t} \nu(T), \qquad \forall\, S \subseteq N. \tag{3}$$

Of course, any set of $2^n$ coefficients $\{m(S)\}_{S \subseteq N}$ does not necessarily correspond to the Möbius transform of a capacity on $N$. The boundary and monotonicity conditions must be ensured [4], i.e., we must have

$$\begin{cases} m(\emptyset) = \emptyset, & \sum_{T \subseteq N} m(T) = 1, \\ \sum_{\substack{T \subseteq S \\ T \ni i}} m(T) \geq 0, & \forall\, S \subseteq N,\ \forall\, i \in S. \end{cases} \tag{4}$$

As shown in [4], in terms of the Möbius representation of a capacity $\mu$ on $N$, for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the Choquet integral of $a$ w.r.t $\mu$ is given by

$$C_{m_\mu}(x) = \sum_{T \subseteq N} m_\mu(T) \bigwedge_{i \in T} x_i, \tag{5}$$

where the symbol $\wedge$ denotes the minimum operator. The notation $C_{m_\mu}$, which is equivalent to the notation $C_\mu$, is used to emphasize the fact that the Choquet integral is here computed w.r.t. the Möbius transform of $\mu$.

### 2.3 Behavioral analysis of the aggregation

The behavior of the Choquet integral as an aggregation operator is generally difficult to understand. For a better comprehension of the interaction phenomena modeled by the underlying capacity, several numerical indices can be computed. The most important ones are the *Shapley value* and the *Shapley interaction indices* ; see e.g [18] for a complete list of behavioral indices.

### 2.4 The concept of $k$-additivity

From the previous results, one can see that a capacity $\mu$ on $N$ is completely defined by the knowledge of $2^n - 2$ coefficients, for instance $\{\mu(S)\}_{\emptyset \neq S \subsetneq N}$ or $\{m_\mu(S)\}_{\emptyset \neq S \subsetneq N}$. Such a complexity may be prohibitive in certain applications. The fundamental notion of $k$-*additivity* proposed by Grabisch [8] enables to find a trade-off between the complexity of the capacity and its modeling ability.

**Definition 2.3** *Let $k \in \{1, \ldots, n\}$. A capacity $\mu$ on $N$ is said to be* $k$-additive *if its Möbius representation satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ such that $t > k$ and there exists at least one subset $T$ of cardinality $k$ such that $m_\mu(T) \neq 0$.*

As one can easily check, the notion of 1-additivity coincides with that of additivity. Let $k \in \{1, \ldots, n\}$ and let $\mu$ be a $k$-additive capacity on $N$. From Eq. (2), we immediately see that a $k$-additive capacity is completely defined by the knowledge of $\sum_{l=1}^{k} \binom{n}{l}$ coefficients.

## 3 The identification problem

Assuming that the utility functions have been determined (see [16]), the next step consists in identifying a capacity, if it exists, such that the Choquet integral w.r.t. this capacity numerically represents the preferences of the DM (see Eq. (1)). When dealing with a real-world MCDA problem, only a finite and usually small subset $\mathcal{O}$ of the set $X$ of objects of interest is generally available. The initial preferences of the DM, from which the capacity is to be determined take at least the form of a partial weak order $\succeq_{\mathcal{O}}$ over $\mathcal{O}$ ; see [14] for a more complete discussion on that subject. In the context of aggregation by the Choquet integral, it seems natural to translate preferential information of the form $x \succ_{\mathcal{O}} x'$ as $C_\mu(u(x)) - C_\mu(u(x')) \geq \delta_C$ where $u(x) := (u_1(x_1), \dots, u_n(x_n))$ and $\delta_C$ is a preference threshold which is to be interpreted as the minimum difference between the overall utilities of two objects which are considered significantly different by the DM.

Most of the identification methods proposed in the literature can be stated under the form of an optimization problem :

$$\text{min or max } \mathcal{F}(\dots)$$
$$\text{subj. to } \begin{cases} \nu(S \cup i) - \nu(S) \geq 0, \forall\, i, \forall\, S \\ \nu(N) = 1, \\ C_\nu(u(x)) - C_\nu(u(x')) \geq \delta_C, \\ \vdots \end{cases}$$

where $\nu$ is a *game* on $N$, i.e. a set function $\nu : 2^N \to \mathbb{R}$ such that $\nu(\emptyset) = 0$ and $\mathcal{F}$ is an objective function that differs w.r.t. to the chosen identification method.

A solution to the above stated optimization problem is a general capacity defined by $2^n - 1$ coefficients. The number of variables involved in it increases exponentially with $n$ and so will the computational time. For large problems, both for computational and simplicity reasons, it may be preferable to restrict the set of possible solutions to $k$-additive capacities, $k \in \{1, \dots, n\}$. The idea is here simply to rewrite the above optimization problem in terms of the Möbius transform of a $k$-additive game using Eqs. (3) and (5), which will decrease the number of variables from $2^n - 1$ to $\sum_{l=1}^{k} \binom{n}{l}$. We obtain

$$\text{min or max } \mathcal{F}(\dots)$$
$$\text{subj. to } \begin{cases} \sum\limits_{\substack{T \subseteq S \\ t \leq k-1}} m_\nu(T \cup i) \geq 0, \; \forall\, i, \forall\, S \\ \sum\limits_{\substack{T \subseteq N \\ 0 < t \leq k}} m_\nu(T) = 1, \\ C_{m_\nu}(u(x)) - C_{m_\nu}(u(x')) \geq \delta_C, \\ \vdots \end{cases}$$

where $m_\nu$ is the Möbius representation of a $k$-additive game $\nu$ on $N$.

Of course, the above optimization problem may be infeasible if the constraints are inconsistent. Such a situation can arise for two reasons : either the preferential information provided by the DM is contradictory, or the number of parameters of the model, i.e. the number of coefficients of the Möbius transform, is too small so that all the constraints can be satisfied. In the latter case, in order to increase the number of free parameters, the approach usually consists in incrementing the order of $k$-additivity. It may happen however that even with a general ($n$-additive) capacity, the constraints imposed by the DM cannot be satisfied. In such a case, the Choquet integral cannot be considered as sufficiently rich for modeling the initial preferences of the DM (see e.g. [27]).

## 4 The main classes of approaches to capacity identification

### 4.1 Least-squares based approaches

Historically, the first approach that has been proposed can be regarded as a generalization of classical multiple linear regression [22]. It requires the additional knowledge of the desired overall evaluations $y(x)$ of the available objects $x \in \mathcal{O}$. The objective function is defined as

$$\mathcal{F}_{LS}(m_\nu) := \sum_{x \in \mathcal{O}} \left[C_{m_\nu}(u(x)) - y(x)\right]^2 .$$

The aim is to minimize the average quadratic distance between the overall utilities computed by means of the Choquet integral $C_{m_\nu}(u(x))$ and the desired overall scores $y(x)$ provided by the DM. The optimization problem takes therefore the form of a quadratic program, not necessarily strictly convex [21], which implies that the solution, if it exits, is not necessarily unique. In order to avoid the use of quadratic solvers, heuristic suboptimal versions of this approach have been proposed by Ishii and Sugeno [12], Mori and Murofushi [22] and Grabisch [7]. In the context of MAUT based on the Choquet integral, the main inconvenient of this approach is that it requires the knowledge of the desired overall utilities $\{y(x)\}_{x \in \mathcal{O}}$, which cannot always be obtained from the DM. Note that a generalization of this approach only requiring a weak order over $\mathcal{O}$ as input information has been recently proposed by Meyer and Roubens [20].

### 4.2 An approach based on linear programming

An approach based on linear programming was proposed by Marichal and Roubens [19]. The proposed identification method can be stated as follows :

$$\max \mathcal{F}_{LP}(\varepsilon) := \varepsilon$$
$$\text{subj. to} \begin{cases} \sum_{\substack{T \subseteq S \\ t \leq k-1}} m_\nu(T \cup i) \geq 0, \; \forall i, \forall S \\ \sum_{\substack{T \subseteq N \\ 0 < t \leq k}} m_\nu(T) = 1, \\ C_{m_\nu}(u(x)) - C_{m_\nu}(u(x')) \geq \delta_C + \varepsilon, \\ \vdots \end{cases}$$

Roughly speaking, the idea of the proposed approach is to maximize the difference between the overall utilities of objects that have been ranked by the DM through the order $\succeq_{\mathcal{O}}$. Indeed, if the DM states that $x \succ_{\mathcal{O}} x'$, he may want the overall utilities to reflect this difference in the most significant way.

The main advantage of this approach is its simplicity. However, similarly to the least-squares based approach presented in the previous subsection, this identification method does not necessarily lead to a unique solution, if any. Furthermore, as it will be illustrated in Section 5, the provided solution can sometimes be considered as too extreme since it corresponds to a capacity that maximizes the difference between overall utilities.

### 4.3 The minimum variance approach

The idea of the minimum variance method [14] is to favor the "least specific" capacity, if any, compatible with the initial preferences of the DM. The objective function $\mathcal{F}_{MV}(m_\nu)$ is defined as the *variance* [14] of the capacity, i.e.

$$\frac{1}{n} \sum_{\substack{i \in N \\ S \subseteq N \setminus i}} \frac{(n-s-1)!s!}{n!} \left( \sum_{T \subseteq S} m_\mu(T \cup i) - \frac{1}{n} \right)^2,$$

As shown in [14], minimizing this variance is equivalent to maximizing the extended Havrda and Charvat entropy of order 2. This method can therefore be equivalently regarded as a maximum entropy approach. The optimization problem takes the form of the following strictly convex quadratic program :

$$\min \mathcal{F}_{MV}(m_\nu)$$
$$\text{subj. to} \begin{cases} \sum_{\substack{T \subseteq S \\ t \leq k-1}} m_\nu(T \cup i) \geq 0, \; \forall i, \forall S \\ \sum_{\substack{T \subseteq N \\ 0 < t \leq k}} m_\nu(T) = 1, \\ C_{m_\nu}(u(x)) - C_{m_\nu}(u(x')) \geq \delta_C, \\ \vdots \end{cases}$$

As discussed in [14], the Choquet integral w.r.t. the minimum variance capacity compatible with the initial preferences of the DM, if it exists, is the one that will exploit the most on average its arguments.

The main advantage of this approach is that it leads to a unique solution, if any, because of the strict convexity of the objective function. In certain cases however the solution can be regarded as "too average".

### 4.4 Practical implementation

The discussed identification methods have been implemented within the Kappalab package [9] for the GNU R statistical system [23]. The package is distributed as free software and can be downloaded from the *Comprehensive R Archive Network* (`http://cran.r-project.org`) or from `http://www.polytech.univ-nantes.fr/kappalab`.

## 5 Applications of the Kappalab R package

### 5.1 Problem description

We consider an extended version of the problem presented in [14] concerning the evaluation of students in an institute training econometricians. The students are evaluated w.r.t. five subjects: statistics (S), probability (P), economics (E), management (M) and English (En). The utilities of seven students $a$, $b$, $c$, $d$, $e$, $f$, $g$ on a $[0, 20]$ scale are given in Table 1.

Table 1: Partial evaluations of the five students.

| Student | S | P | E | M | En | Mean |
|---|---|---|---|---|---|---|
| $a$ | 18 | 11 | 11 | 11 | 18 | 13.80 |
| $b$ | 18 | 11 | 18 | 11 | 11 | 13.80 |
| $c$ | 11 | 11 | 18 | 11 | 18 | 13.80 |
| $d$ | 18 | 18 | 11 | 11 | 11 | 13.80 |
| $e$ | 11 | 11 | 18 | 18 | 11 | 13.80 |
| $f$ | 11 | 11 | 18 | 11 | 11 | 12.40 |
| $g$ | 11 | 11 | 11 | 11 | 18 | 12.40 |

Assume that the institute is slightly more oriented towards statistics and probability and suppose that the DM considers that there are 3 groups of subjects: statistics and probability, economics and management, and English. Furthermore, he/she considers that within the two first groups, subjects are somewhat substitutive, i.e. they overlap to a certain extent. Finally, if a student is good in statistics or probability (resp. bad in statistics and probability), it is better that he/she is good in English (resp. economics or management) rather than in economics or management (resp. English). This reasoning leads to the following ranking:

$$a \succ_{\mathcal{O}} b \succ_{\mathcal{O}} c \succ_{\mathcal{O}} d \succ_{\mathcal{O}} e \succ_{\mathcal{O}} f \succ_{\mathcal{O}} g. \qquad (6)$$

We shall further assume that the DM considers that two students are significantly different if their overall utilities differ by at least half a unit.

By considering students $a$ and $b$, and $f$ and $g$, it is easy to see that the criteria do not satisfy mutual preferential independence, which implies that there is no additive model that can numerically represent the above weak order.

In order to use the identification methods overviewed in the previous section and implemented in Kappalab, we first create 7 R vectors representing the students:

```
> a <- c(18,11,11,11,18)
> b <- c(18,11,18,11,11)
> c <- c(11,11,18,11,18)
> d <- c(18,18,11,11,11)
> e <- c(11,11,18,18,11)
> f <- c(11,11,18,11,11)
> g <- c(11,11,11,11,18)
```

The symbol `>` represents the prompt in the R shell, the symbol `<-` the assignment operator, and `c` is the R function for vector creation.

### 5.2 The least squares approach

In order to apply the least squares approaches presented in Subsection 4.1, the 7 vectors previously defined and representing the students need first to be concatenated into a 7 row matrix, called `C` here, using the `rbind` ("row bind") matrix creation function:

```
> C <- rbind(a,b,c,d,e,f,g)
```

Then, the DM needs to provide overall utilities for the seven students. Although it is unrealistic to consider that this information can always be given, we assume in this subsection that the DM is able to provide it. He respectively assigns 15, 14.5, 14, 13.5, 13, 12.5 and 12 to $a$, $b$, $c$, $d$, $e$, $f$ and $g$. These desired overall utilities are encoded into a 7 element R vector:

```
> overall <- c(15,14.5,14,13.5,13,
               12.5,12)
```

The least squares identification routine based on quadratic programming (providing an optimal but not necessarily unique solution) can then be called by typing:

```
> ls <- least.squares.capa.ident(5,2,
            C,overall)
```

in the R terminal. The first argument sets the number of criteria, the second fixes the desired order of $k$-additivity, and the two last represent the matrix containing the partial utilities and the vector containing the desired overall utilities respectively. The result is stored in an R list object, called here `ls`, containing all the relevant information for analyzing the results.

The solution, a 2-additive capacity given under the form of its Möbius representation, can be obtained by typing:

```
> m <- ls$solution
```

and visualized by entering `m` in the R terminal:

```
> m
                    Mobius.capacity
{}                  0.000000
{1}                 0.311650
{2}                 0.176033
...                 ...
{4,5}               0.001752
```

As discussed in Subsection 4.1, for the considered example, the obtained solution is probably not unique [21].

The Choquet integral for instance of $a$ w.r.t. the solution can be obtained by typing:

```
> Choquet.integral(m,a)
[1] 15
```

The overall utilities computed using the Choquet integral w.r.t. the 2-additive solution are given in the last column of the table below, the sixth column containing the desired overall evaluations:

| | S | P | E | M | En | Given | Mean | LS |
|---|---|---|---|---|---|---|---|---|
| a | 18 | 11 | 11 | 11 | 18 | 15.0 | 13.8 | 15.0 |
| b | 18 | 11 | 18 | 11 | 11 | 14.5 | 13.8 | 14.5 |
| c | 11 | 11 | 18 | 11 | 18 | 14.0 | 13.8 | 14.0 |
| d | 18 | 18 | 11 | 11 | 11 | 13.5 | 13.8 | 13.5 |
| e | 11 | 11 | 18 | 18 | 11 | 13.0 | 13.8 | 13.0 |
| f | 11 | 11 | 18 | 11 | 11 | 12.5 | 12.4 | 12.5 |
| g | 11 | 11 | 11 | 11 | 18 | 12.0 | 12.4 | 12.0 |

As one can see, this approach enables to recover the overall utilities provided by the DM.

The Shapley value and the interaction indices of the 2-additive solution can be computed by typing `Shapley.value(m)` and `interaction.indices(m)` in the R terminal.

### 5.3 The LP and minimum variance approaches

As discussed earlier, the least squares approach applied in the previous subsection is not well adapted to MAUT since it relies on information that a DM cannot always provide. The LP and the minimum variance approaches require only a partial weak order over the available objects, such as the one provided by the DM in Eq. (6). This weak order is naturally translated as

$$C_{m_\nu}(a) > C_{m_\nu}(b) > C_{m_\nu}(c) > C_{m_\nu}(d) \\ > C_{m_\nu}(e) > C_{m_\nu}(f) > C_{m_\nu}(g),$$

with preference threshold $\delta_C = 0.5$.

Practically, the preference threshold is stored in an R variable:

```
> delta.C <- 0.5
```

and the weak order over the students is encoded into a 6 row R matrix:

```
> Acp <- rbind(c(a,b,delta.C),
               c(b,c,delta.C),
               c(c,d,delta.C),
               c(d,e,delta.C),
               c(e,f,delta.C),
               c(f,g,delta.C))
```

each row containing a constraint of the form $C_{m_\nu}(u(x)) \geq C_{m_\nu}(u(y)) + \delta_C$.

The LP approach is then invoked by typing:

```
> lp <- lin.prog.capa.ident(5,2,
             A.Choquet.preorder = Acp)
```

The first argument fixes the number of criteria, the second sets the desired order of $k$-additivity for the solution, and the last contains the partial weak order provided by the DM. All the relevant information to analyze the solution is stored in the R object `lp`.

The minimum variance approach is called similarly:

```
> mv <- mini.var.capa.ident(5,2,
           A.Choquet.preorder = Acp)
```

The overall utilities computed using the Choquet integral w.r.t. the 2-additive solutions are given in the following table:

| | S | P | E | M | En | Mean | LP | MV |
|---|---|---|---|---|---|---|---|---|
| a | 18 | 11 | 11 | 11 | 18 | 13.8 | 18.00 | 15.25 |
| b | 18 | 11 | 18 | 11 | 11 | 13.8 | 17.36 | 14.75 |
| c | 11 | 11 | 18 | 11 | 18 | 13.8 | 16.73 | 14.25 |
| d | 18 | 18 | 11 | 11 | 11 | 13.8 | 16.09 | 13.75 |
| e | 11 | 11 | 18 | 18 | 11 | 13.8 | 15.45 | 13.25 |
| f | 11 | 11 | 18 | 11 | 11 | 12.4 | 14.82 | 12.75 |
| g | 11 | 11 | 11 | 11 | 18 | 12.4 | 14.18 | 12.25 |

Note that, as expected, the LP approach leads to more dispersed utilities, which reach the maximum value (18) that a Choquet integral can take for the seven students. Note also that, for the minimum variance approach, the differences between the overall utilities of two consecutive students in the weak order provided by the DM equal exactly $\delta_C$. This last observation follows from the fact that, in this example, the aim this methods is roughly to find the Choquet integral that is the closest to the simple arithmetic mean while being in accordance with the preferential information provided by the DM.

The Shapley values of the 2-additive solutions are:

| | S | P | E | M | En |
|---|---|---|---|---|---|
| LP | 0.45 | 0.00 | 0.27 | 0.05 | 0.23 |
| MV | 0.27 | 0.16 | 0.21 | 0.14 | 0.22 |

As one can see, all three solutions designate statistics (S) as the most important criterion. Note that the LP solution is very extreme, since the overall importance of probability (P) and management (M) is very small and that of S is close to one half. Remark that the overall importances of the criteria are not in accordance with the orientation of the institution. Indeed, one would have expected to obtain that statistics (S) and probability (P), and economics (E) and management (M), have the same importances. This is due to the small number of students ranked by the DM and can been fixed by imposing additional constraints.

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